PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The Structure of Free Vertical Shear Layers in a Rotating Fluid and the Motion Produced by a Slowly Rising Body

D. W. Moore and P. G. Saffman

Phil. Trans. R. Soc. Lond. A 1969 **264**, 597-634
doi: 10.1098/rsta.1969.0036

THE STRUCTURE OF FREE VERTICAL SHEAR LAYERS IN A ROTATING FLUID AND THE MOTION PRODUCED BY A SLOWLY RISING BODY

By D. W. MOORE† and P. G. SAFFMAN

California Institute of Technology, Pasadena, California 91109

(*Communicated by K. Stewartson, F.R.S.—Received* 28 *October* 1968)

CONTENTS

This paper considers axial motion in a rapidly rotating fluid of small viscosity. It is shown that solutions for the structure of vertical free shear layers must be allowed to be singular at the points where they receive fluid from a rising or spinning axisymmetric body. The possible types of singularities are elucidated by the use of similarity solutions and an hypothesis is introduced to limit their strength. Three particular cases of axially bounded motion are considered in detail; the split disk, the rising disk and the rising sphere. The hypothesis is shown to lead to a unique solution for the Stewartson layers. For the rising body, a Wiener–Hopf problem, which is independent of the body shape, must be solved for the central part of the Stewartson layers.

1. Introduction

When a body is moved slowly in any manner in a rapidly rotating liquid the boundary of the Taylor column will be a detached viscous shear layer. By 'slowly' we mean that the Rossby number $U/L\Omega$ is small and by 'rapidly rotating' we mean that the Ekman number $E = \nu/L^2\Omega$ is small, where U is the velocity scale, L the scale of the body in a direction perpendicular to the rotation $\mathbf{\Omega}$—we term such a direction lateral—and ν is the kinematic viscosity of the liquid.

We are interested in this paper in the shear layers which form when a solid of revolution moves with its axis of symmetry parallel to $\mathbf{\Omega}$ and with the velocity of its centre parallel to $\mathbf{\Omega}$. The Taylor column is then a circular cylinder touching the body of revolution at its equator.

† Permanent address: Department of Mathematics, Imperial College, London, S.W. 7.

Three methods have been used to discuss the shear layers formed in this circumstance.

The first method, used by Morrison & Morgan (1956) and by Stewartson (1957) is to take the limit $E \to 0$ in the exact solution of the full linearized Navier–Stokes equations

$$2\mathbf{\Omega} \wedge \mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} \tag{1.1}$$

and

$$\operatorname{div} \mathbf{u} = 0. \tag{1.2}$$

Morrison & Morgan were interested in axially unbounded flow. Stewartson found the exact solution for the split disk configuration described in §4. He proved that for this problem the detached shear layer had a sandwich structure, a central shear layer of thickness $E^{\frac{1}{3}}$ being sandwiched between fatter layers of thickness $E^{\frac{1}{4}}$. The $\frac{1}{4}$ layers were associated with a large gradient of swirl velocity, which was continuous across the $\frac{1}{3}$ layer. The $\frac{1}{3}$ layer was characterized by intense axial motion and carried as much volume flux as the geostrophic interior of the Taylor column. This flow has been recently studied experimentally by Baker (1967) and Stewartson's predictions have been confirmed.

Only problems in cylindrical geometry for which the exact solution of (1.1) and (1.2) can be found by separation of variables have so far been studied in this way. However, this method avoids the uncertainties (not always recognized) inherent in the more powerful methods to be described.

Carrier (1964) has described a method of treating the shear layer which is not confined to simple geometries. This is to approximate to (1.1) by neglecting derivatives parallel to the Taylor column boundary. This approximation fails in the geostrophic interior, but there the viscous term is in any case negligible. Thus the resulting considerably simpler equations are valid everywhere outside the Ekman layers on the body surface and on the ends of the container. The approximate equations are second order in the derivative parallel to $\mathbf{\Omega}$. Now as was pointed out by Greenspan & Howard (1963) and by Jacobs (1964) the Ekman condition remains valid so long as lateral rates of change are $o(\nu^{-\frac{1}{2}})$.† Since any straight line in the shear layer parallel to $\mathbf{\Omega}$ will meet the Ekman layer system in two points, the Ekman condition provides sufficient boundary conditions. Thus one gets the geostrophic interior and the shear layer sandwich in one go.

The different thicknesses and dynamical balances of the $\frac{1}{3}$ layer and the $\frac{1}{4}$ layers suggests that one should go farther and use equations appropriate to the separate layers. This is the third method tried first by Stewartson (1966)‡ in his study of the shear layers generated by the co-axial rotation of concentric spherical surfaces. The inner and outer layers satisfy ordinary differential equations and could thus, if one had sufficient boundary conditions, be discussed without difficulty. Unfortunately this is not so and one is forced to seek the missing conditions by asking what restrictions the dynamics of the central $\frac{1}{3}$ layer place on the velocity field at its inner and outer edges. A systematic way to find these restrictions, which emerge as jump conditions on the swirl velocity, was described by Stewartson.

The object of our work is to extend Stewartson's work to the problem of a solid revolution rising in a container (Moore & Saffman 1968; Maxworthy 1968). We do this in three stages,

† Rather than express this and subsequent similar estimates in terms of E, U and L we use this convenient shorthand which, since E is usually the only small parameter, will not cause ambiguity.

‡ But partially anticipated by Proudman (1956).

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

in §§ 4, 5, 6, where progressively harder problems are studied. The advantage and power of Stewartson's method becomes apparent when one sees that, while the structure of the $\frac{1}{4}$ layers varies considerably from case to case—their thickness is not the same and they can even disappear completely†—the structure of the $\frac{1}{3}$ layer is virtually unchanged. In particular the $\frac{1}{3}$ layer generated by a rising sphere is *identical* (apart from multiplicative scale factors) with that generated by a rising disk of the same radius. The $\frac{1}{3}$ layer does not, to leading order, 'see' the body at all, the body shape being apparent only in the structure of the $\frac{1}{4}$ layers. All it does see is an annular region at the equator of the body where the Ekman layer on the body is separating from the surface. This joining region will be on a lateral scale smaller than $\nu^{\frac{1}{3}}$ and so will appear as a *singularity* in the $\frac{1}{3}$ layer solution.

Mathematically, we have to solve a Wiener–Hopf problem—though the solution of a single such problem, which is given in detail in § 5, gets the structure of the $\frac{1}{3}$ layer for all rising bodies of revolution, whereas the second method would involve the discussion of a different, more difficult such problem in each case. As is usual in a Wiener–Hopf problem, one must specify the strength of the singularity at the sharp edge before the solution can be found. This is the real problem which has to be faced in extending Stewartson's method.

Information about the singularity should really be sought by working on a finer scale, that is, by examining the dynamics of the joining region itself. We have not been able to do this and we have instead approached the problem indirectly. In § 3 we look at what sorts of singularity the solutions of the $\frac{1}{3}$ layer equations can have and we pick out the ones which can occur at a sharp edge. There is an infinite sequence of possible singularities in which the axial velocity in the $\frac{1}{3}$ layer goes to infinity like $z^{-\frac{1}{6}}, z^{-\frac{1}{3}}, z^{-\frac{1}{2}}, \ldots$, where z is axial distance from the singularity. All these singularities are consistent with the $\frac{1}{3}$ layer equations and with the boundary conditions and we have been forced to impose an *ad hoc* condition on the radial pressure gradient to limit the strength of the singularity which can arise. We insist that the radial pressure gradient be not larger inside the $\frac{1}{3}$ layer than it is just outside, so our condition is like the Kutta–Joukowsky condition of airfoil theory.

In § 7, we consider the case where the axial separations of the boundaries are so large that the $\frac{1}{4}$ layers have spread viscous effects throughout the Taylor column. Finally in § 8, we re-derive the solution in the unbounded case and discuss why boundary-layer methods are not adequate.

2. The shear-layer equations

Let x, y, z be orthogonal coordinates, such that the surface $x = 0$ is the Taylor column boundary and the surfaces $z =$ constant are planes perpendicular to the axis of rotation. Then if (u, v, w) is the velocity of the liquid at (x, y, z) and p is the reduced pressure divided by the density, the shear layer equations are

$$-2\Omega v = -\frac{1}{h_1^{(0)}}\frac{\partial p}{\partial x} + \frac{\nu}{h_1^{(0)2}}\frac{\partial^2 u}{\partial x^2} \tag{2.1}$$

† The breadth of the inner layer depends on the shape of the solid of revolution near its equator. If in cylindrical polars $z \sim (r-a)^n$, where a is the radius of the equator and where $0 < n < 1$, the inner layer has thickness $\nu^{1/(n+3)}$. For a lenticular body, $n = 1$ and we get a $\frac{1}{4}$ layer. For a body of finite curvature like a sphere, $n = \frac{1}{2}$ and the inner layer is of thickness $\nu^{\frac{2}{7}}$.

$$2\Omega u = -\frac{1}{h_2^{(0)}}\frac{\partial p}{\partial y} + \frac{\nu}{h_1^{(0)2}}\frac{\partial^2 v}{\partial x^2} \tag{2.2}$$

$$0 = -\frac{\partial p}{\partial z} + \frac{\nu}{h_1^{(0)2}}\frac{\partial^2 w}{\partial x^2} \tag{2.3}$$

and

$$h_2^{(0)}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}(h_1^{(0)}v) + h_1^{(0)}h_2^{(0)}\frac{\partial w}{\partial z} = 0, \tag{2.4}$$

where $h_1(x,y)$, $h_2(x,y)$ and $h_3 = 1$ are the line elements and where

$$h_1^{(0)} = h_1(0,y) \tag{2.5}$$

and

$$h_2^{(0)} = h_2(0,y). \tag{2.6}$$

Now the viscous term in (2.1) can be shown to be negligible. Let $\delta(\nu)$ be the shear-layer thickness, so that $\partial/\partial x \sim 1/\delta$ and assume that $\partial/\partial y \sim 1/L$, the lateral scale. Then from (2.2)

$$u \sim \frac{p}{L\Omega} + \frac{\nu v}{\Omega\delta^2}. \tag{2.7}$$

Thus the viscous term in (2.1) is

$$\frac{p\nu}{L\Omega\delta^2} + \frac{\nu^2 v}{\Omega\delta^4}$$

so that this is negligible provided that

$$\delta \gg \max\left(\frac{\nu}{L\Omega}, \frac{\nu^{\frac{1}{2}}}{\Omega^{\frac{1}{2}}}\right). \tag{2.8}$$

Since, by hypothesis

$$E = \nu/L^2\Omega \ll 1 \tag{2.9}$$

it is negligible provided that

$$\delta \gg \nu^{\frac{1}{2}}/\Omega^{\frac{1}{2}}. \tag{2.10}$$

This inequality will be satisfied if the shear layer is much thicker than the Ekman layer, which will prove to be the case.

If we drop the viscous term in (2.1) and eliminate u and p from the resulting equations, we find that

$$\frac{\nu}{h_1^{(0)3}}\frac{\partial^3 w}{\partial x^3} = 2\Omega\frac{\partial v}{\partial z} \tag{2.11}$$

and

$$\frac{\nu}{h_1^{(0)3}}\frac{\partial^3 v}{\partial x^3} = -2\Omega\frac{\partial w}{\partial z}. \tag{2.12}$$

In deriving equations (2.1) to (2.4) from the Navier–Stokes equations, we have neglected contributions to the viscous forces from first derivatives with respect to x and we have neglected the variation of line elements across the shear layer. The consequent error in (2.11) and (2.12) is $O(\delta/L)$ and is at most as great as the relative error arising from the approximation to (1).

If in both (2.11) and (2.12), the two sides of the equation are comparable, then we must have $v/w \sim 1$ and $\delta \sim \nu^{\frac{1}{3}}$. This is the $\frac{1}{3}$ layer. If, as in the inner and outer layers $v \gg w$, then

$$0 = \partial v/\partial z \tag{2.13}$$

and

$$\frac{\nu}{h_1^{(0)3}}\frac{\partial^3 v}{\partial x^3} = -2\Omega\frac{\partial w}{\partial z} \tag{2.14}$$

and clearly

$$\delta \gg \nu^{\frac{1}{3}}.$$

In the axi-symmetric problems discussed in detail we use cylindrical polar coordinates (r, θ, z). The axis Oz is parallel to Ω and coincides with the axis of symmetry of the body of revolution. The equator of the body is $r = a$, $z = 0$ and we write $x = r - a$. Then (2.11) and (2.12) become

$$\nu \frac{\partial^3 w}{\partial x^3} = 2\Omega \frac{\partial v}{\partial z} \tag{2.15}$$

and

$$\nu \frac{\partial^3 v}{\partial x^3} = -2\Omega \frac{\partial w}{\partial z}. \tag{2.16}$$

We shall use lower case letters for quantities in the $\frac{1}{3}$ layer and corresponding capital letters for the quantities in the $\frac{1}{4}$ layers and in the geostrophic region, since the approximation (2.13) makes the swirl velocity there a function of x alone.

3. Similarity solutions

The velocity field in the $\frac{1}{3}$ layer satisfies the equation

$$\frac{\partial^3 \chi}{\partial x^3} = -\frac{2\Omega}{\nu} \mathrm{i} \frac{\partial \chi}{\partial z}, \tag{3.1}$$

where

$$\chi = w + \mathrm{i}v. \tag{3.2}$$

Our objective in this section is a systematic classification of the similarity solutions of (3.1). Let us introduce a similarity variable τ, clearly suggested by the form of (3.1), defined for all $z (> 0)$ by

$$\tau = x\left(\frac{2\Omega}{\nu z}\right)^{\frac{1}{3}} = \eta\left(\frac{2\Omega}{z}\right)^{\frac{1}{3}}. \tag{3.3}$$

The existence of similarity solutions involving τ was noticed by Morrison & Morgan (1956). For $z < 0$, we construct solutions from the fact that $\chi^*(x, -z)$ is a solution of (3.1) if $\chi(x, z)$ is.

We can verify that, for any real constant m,

$$\chi_m(x, \eta) = z^m h(\tau) \quad (z > 0), \tag{3.4}$$

is a solution of (3.1) provided that h_m satisfies the ordinary differential equation

$$L_m h_m \equiv \frac{\mathrm{d}^3 h_m}{\mathrm{d}\tau^3} - \tfrac{1}{3}\mathrm{i}\tau \frac{\mathrm{d}h_m}{\mathrm{d}\tau} + \mathrm{i}m h_m = 0. \tag{3.5}$$

The general solution of (3.5) is obtained as a contour integral by Laplace's method and is

$$h_m(\tau) = A \int_a^b \mathrm{e}^{-\mathrm{i}p\tau}\, \mathrm{e}^{-p^3} p^{-3m-1}\, \mathrm{d}p, \tag{3.6}$$

where A is a complex constant. The end-points of the path of integration in the p plane must be chosen so that

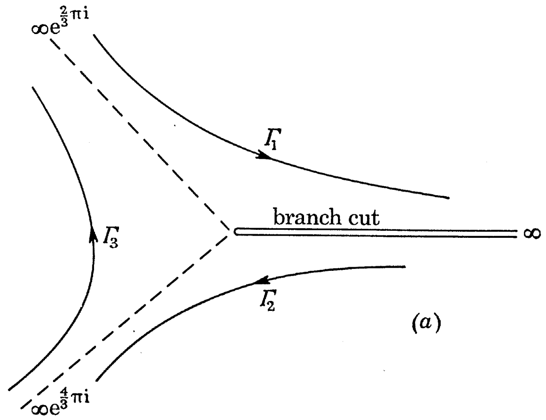
$$[p^{-3m}\, \mathrm{e}^{-p^3}\, \mathrm{e}^{-p\mathrm{i}\tau}]_a^b = 0 \quad \text{for all } \tau, \tag{3.7}$$

and, unless $3m$ is integral, the p plane must be suitably cut.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

If $m \geqslant 0$ possible values for a and b are ∞, $\infty\,e^{\frac{2}{3}\pi i}$ and $\infty\,e^{\frac{4}{3}\pi i}$ and in figure 1 we sketch three possible Γ_1, Γ_2, and Γ_3, giving, because of the cut, three linearly independent solutions of (3.5).†

When $m < 0$ we can choose $a = 0$ as the starting-point of the contour and we get the three contours shown in figure 1.



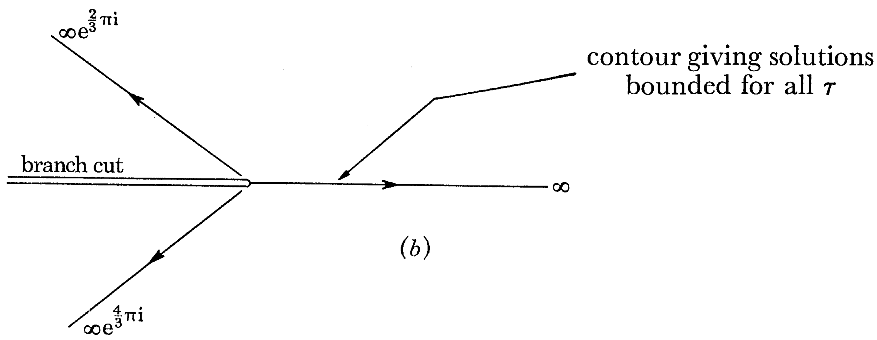


FIGURE 1. (*a*) Contours in the complex p plane which give solutions of (3.5) when $m \geqslant 0$. (*b*) Contours when $m < 0$.

In this report we are interested in detached shear layers, which means that τ can range from $-\infty$ to $+\infty$. The requirement that h is to be bounded as $\tau \to \pm\infty$ places a severe restriction on the contour used in (3.6) since, unless p is real at every point of the contour, the integrand will become exponentially large as either $\tau \to +\infty$ or $\tau \to -\infty$. No choice of contour will satisfy this requirement when $m \geqslant 0$ and for $m < 0$ the only possible contour is the positive real axis. Thus we have shown that

(*a*) Equation (5) has no solutions which are bounded in the entire range $-\infty < \tau < \infty$ when $m \geqslant 0$.

† If $3m$ is integral the cut disappears and the three solutions satisfy a linear relation obtained by applying Cauchy's theorem to the contour obtained by joining the three contours Γ_i at their extremities. This is obvious when $m = -\frac{1}{3}$, for which value (3.5) is really only second order. We can express the solutions of (3.5) for other values of m which make $3m$ integral as derivatives or integrals of the two independent solutions when $3m = -1$ by using the result that $dh_m/d\tau$ satisfies $L_{m-\frac{1}{3}}\, dh_m/d\tau = 0$.

(*b*) When $m<0$ the unique solution of (3.5) bounded in $-\infty<\tau<\infty$ is a multiple of

$$H_m(\tau)=\int_0^\infty e^{-ip\tau}\,e^{-p^3}\,p^{-3m-1}\,dp. \tag{3.8}$$

The asymptotic behaviour of (3.8) as $\tau\to\pm\infty$ is easily shown to be

$$H_m\sim\tau^{3m}\,e^{\frac{3}{2}m\pi i}\,(-3m-1)! \quad \text{as} \quad \tau\to\infty \tag{3.9}$$

and
$$H_m\sim(-\tau)^{3m}\,e^{-\frac{3}{2}m\pi i}\,(-3m-1)! \quad \text{as} \quad \tau\to-\infty. \tag{3.10}$$

Thus
$$\chi_m\sim\eta^{3m}(2\Omega)^m\,e^{\frac{3}{2}m\pi i}\,(-3m-1)! \quad \text{as} \quad \tau\to\infty \tag{3.11}$$

and
$$\chi_m\sim(-\eta)^{3m}\,(2\Omega)^m\,e^{-\frac{3}{2}m\pi i}\,(-3m-1)! \quad \text{as} \quad \tau\to-\infty. \tag{3.12}$$

Note that the asymptotic value of χ_m as $\tau\to\pm\infty$ is independent of z and that the decay is only algebraic. This will have important consequences in infinite regions where the $\frac{1}{3}$ layers have a similarity structure everywhere, and not just near the joining region.

We can use (3.11) and (3.12) to determine the behaviour of the similarity solutions on the plane $z=0$. Suppose $Az^mH_m(\tau)$ is such a solution, where $A=|A|\,e^{i\beta}$. Then on $z=0+$ we have

$$\left.\begin{aligned} w&\sim C\eta^{3m}\cos\left(\tfrac{3}{2}m\pi+\beta\right) \quad (\eta>0)\\ v&\sim C\eta^{3m}\sin\left(\tfrac{3}{2}m\pi+\beta\right) \quad (\eta>0)\end{aligned}\right\} \tag{3.13}$$

and

$$\left.\begin{aligned} w&\sim C(-\eta)^{3m}\cos\left(-\tfrac{3}{2}m\pi+\beta\right) \quad (\eta<0)\\ v&\sim C(-\eta)^{3m}\sin\left(-\tfrac{3}{2}m\pi+\beta\right) \quad (\eta<0),\end{aligned}\right\} \tag{3.14}$$

where $C=|A|\,(2\Omega)^m\,(-3m-1)!$. The change of phase in the trigonometric factor is a consequence of the fact that, in general, w and v have no symmetries about $\eta=0$.

A situation which we will encounter in specific problems is as follows. The half plane† $z=0$, $\eta<0$ is that portion of the surface of a moving body inside the $\frac{1}{3}$ layer. Consideration of the Ekman suction equation shows that, in general, a swirl velocity v in the $\frac{1}{3}$ layer requires an inflow $o(v)$ (we give precise results in later sections) so that since $w=O(v)$ in the $\frac{1}{3}$ layer, we must have

$$w=0 \quad \text{on} \quad z=0 \quad (\eta<0). \tag{3.15}$$

The velocity field in the $\frac{1}{3}$ layer is analytic away from the body so that, in particular,

$$w,v \quad \text{are continuous at} \quad z=0 \quad (\eta>0). \tag{3.16}$$

If $\chi_m(\eta,z)$ is a solution of (3.1), so is $\chi_m^*(\eta,-z)$ and thus for $z<0$ the general similarity solution is of the form $A'(-z)^m H_m^*(\tau')$ where $\tau'=\eta(2\Omega/-z)^{\frac{1}{3}}$. Thus on $z=0-$, we find from the asymptotic expansions that

$$\left.\begin{aligned} w&=C'\eta^{3m}\cos\left(-\tfrac{3}{2}m\pi+\beta'\right) \quad (\eta>0)\\ v&=C'\eta^{3m}\sin\left(-\tfrac{3}{2}m\pi+\beta'\right) \quad (\eta>0)\end{aligned}\right\} \tag{3.13$'$}$$

and

$$\left.\begin{aligned} w&=C'(-\eta)^{3m}\cos\left(\tfrac{3}{2}m\pi+\beta'\right) \quad (\eta<0)\\ v&=C'(-\eta)^{3m}\sin\left(\tfrac{3}{2}m\pi+\beta'\right) \quad (\eta<0).\end{aligned}\right\} \tag{3.14$'$}$$

The boundary condition (3.15) shows that

$$\cos\left(-\tfrac{3}{2}m\pi+\beta\right)=0, \quad \cos\left(\tfrac{3}{2}m\pi+\beta'\right)=0, \tag{3.17}$$

† We shall see in §6 that even for a sphere the portion of it inside the $\frac{1}{3}$ layer can be regarded, at least to leading order, as a plane perpendicular to the axis of rotation.

while condition (3.16) gives $C = C'$ and

$$\tfrac{3}{2}m\pi + \beta = -\tfrac{3}{2}m\pi + \beta' + 2n\pi \tag{3.18}$$

where n is an integer. Hence $m = \tfrac{1}{6}p$, p integral (3.19)

so that since $m < 0$ for a solution bounded for all τ

$$m = -\tfrac{1}{6}, -\tfrac{1}{3}, -\tfrac{1}{2}, \ldots. \tag{3.20}$$

Thus only certain values of m and consequently only certain singularities can arise for boundary conditions (3.15) and (3.16).

It can be verified that if $m = -\tfrac{1}{3}, -\tfrac{2}{3}, \ldots$

$$w = 0 \quad \text{on} \quad z = 0 \quad (\eta > 0) \tag{3.21}$$

and that w is antisymmetrical and v symmetrical about $z = 0$; while if $m = -\tfrac{1}{6}, -\tfrac{1}{2}, \ldots$

$$v = 0 \quad \text{on} \quad z = 0 \quad (\eta > 0) \tag{3.22}$$

and w is symmetrical and v antisymmetrical. Sometimes (3.21) or (3.22) emerge as requirements by virtue of symmetries in the problem. The 'split-disk' studied in § 4 is a case where (3.21) must hold. On the other hand, for a disk rising without rotation in an unbounded fluid, the symmetry can be shown to imply (3.22). Indeed, the similarity solution with $m = -\tfrac{1}{6}$ was shown to arise in this problem by Morrison & Morgan (1956).

The particular solution for $m = -\tfrac{1}{3}$, which is

$$\chi = (2\Omega)^{\frac{1}{3}}\, Cz^{-\frac{1}{3}} \int_0^\infty \mathrm{e}^{-\mathrm{i}p\tau}\, \mathrm{e}^{-p^3}\, \mathrm{d}p \tag{3.23}$$

$$= (2\Omega)^{\frac{1}{3}}\, CR(\eta, z) \quad \text{say,} \tag{3.24}$$

is of special interest. By a change of variable

$$R(\eta, z) = \frac{1}{(2\Omega)^{\frac{1}{3}}} \int_0^\infty \mathrm{e}^{-\mathrm{i}\alpha\eta}\, \mathrm{e}^{-z\alpha^3/2\Omega}\, \mathrm{d}\alpha. \tag{3.25}$$

Thus

$$\int_{-\infty}^{\infty} R(\eta, z)\, \mathrm{d}\eta = \frac{1}{(2\Omega)^{\frac{1}{3}}} \int_0^\infty 2\pi\delta(\alpha)\, \mathrm{e}^{-z\alpha^3/2\Omega}\, \mathrm{d}\alpha = \frac{\pi}{(2\Omega)^{\frac{1}{3}}}, \tag{3.26}$$

so that the vertical volume flux per unit length in this shear layer is independent of z. Thus the solution (3.23) represents the motion due to a line source of strength $2\pi C\nu^{\frac{1}{3}}$ per unit length located at $\eta = 0$, $z = 0$. The fluid for the source is provided by mass flux in the two Ekman layers on $z = 0\pm$, $\eta < 0$, which are of zero thickness with the scaling that leads to (3.1).

The similarity solutions with other allowable values of m can be shown easily to have zero vertical volume flux across a plane $z =$ constant, with the exception of $m = -\tfrac{1}{6}$ which has a divergent volume flux because of the slow $\eta^{-\frac{1}{2}}$ decay. For this reason, we do not expect the $m = -\tfrac{1}{6}$ solution to occur to leading order in problems amenable to a boundary-layer type analysis, and we shall see later that it is in fact excluded at least to leading order by the 'Kutta condition' mentioned in § 1 for flows which are bounded in the vertical direction. Unbounded flows cannot be solved by methods utilizing boundary-layer type approximations (see § 8).

4. The split-disk problem

The first problem we will examine in detail is one solved by Stewartson (1957). The rigid planes $z = 0$ and $z = h$ have concentric circular central portions, each of radius a, which can be rotated at different angular velocities from the rest of the plane. Suppose both outer portions rotate with angular velocity Ω, while the central disks $r < a$, $z = 0, h$ have slightly different angular velocities $\Omega(1+\epsilon')$, $\Omega(1+\epsilon)$ respectively. The problem is to find the motion driven in viscous fluid filling the space between the plates.

Let $(U_G(r), V_G(r), W_G(r))$ be the geostrophic interior. Since $p = p(r)$, the tangential component of the geostrophic flow equations gives

$$U_G(r) = 0. \tag{4.1}$$

There is a radial flux inwards through the Ekman layers on the upper and lower plates of amount, per unit length of circumference,

$$\tfrac{1}{2}(\nu/\Omega)^{\frac{1}{2}}\,(V_G-\epsilon\Omega r), \quad \tfrac{1}{2}(\nu/\Omega)^{\frac{1}{2}}\,(V_G-\epsilon'\Omega r). \tag{4.2}$$

We can now find the geostrophic interior by demanding that the net radial flux in the two Ekman layers is zero, that is

$$\tfrac{1}{2}(\nu/\Omega)^{\frac{1}{2}}\,(V_G-\epsilon'\Omega r)+\tfrac{1}{2}(\nu/\Omega)^{\frac{1}{2}}\,(V_G-\epsilon\Omega r) = 0 \quad (0 \leqslant r < a), \tag{4.3}$$

so that

$$V_G(r) = \tfrac{1}{2}(\epsilon+\epsilon')\,\Omega r \quad (0 \leqslant r < a). \tag{4.4}$$

Thus the core of the Taylor column rotates with the mean angular velocity of the two disks.

By continuity

$$W_G(r) = \tfrac{1}{2}(\nu\Omega)^{\frac{1}{2}}\,(\epsilon-\epsilon') \quad (0 \leqslant r < a), \tag{4.5}$$

so that there is an $O(\nu^{\frac{1}{2}})$ axial drift from the slower to the faster disk. For $r > a$, the same argument gives

$$U_G(r) = V_G(r) = W_G(r) = 0 \quad (r > a). \tag{4.6}$$

Thus, we have solid body rotation for $r > a$.

Both $V_G(r)$ and $W_G(r)$ are discontinuous on the Taylor column boundary $r = a$ and the real problem is to see how viscous forces smooth out the discontinuity.

We first remark that $W_G/V_G = O(\nu^{\frac{1}{2}})$ so that the outer portions of the shear layer must be of the second type discussed in § 2. Thus

$$\frac{\mathrm{d}V}{\mathrm{d}z} = 0 \quad \text{and} \quad U = \frac{\nu}{2\Omega}\frac{\mathrm{d}^2V}{\mathrm{d}x^2}, \tag{4.7}$$

where

$$x = r-a \tag{4.8}$$

The radial flux condition now becomes, for $x < 0$,

$$-\tfrac{1}{2}(\nu/\Omega)^{\frac{1}{2}}\,(V-\epsilon'\Omega a)-\tfrac{1}{2}(\nu/\Omega)^{\frac{1}{2}}\,(V-\epsilon\Omega a)+\frac{\nu h}{2\Omega}\frac{\mathrm{d}^2V}{\mathrm{d}x^2} = 0, \tag{4.9}$$

and the solution which matches smoothly to the geostrophic interior as $x/\nu^{\frac{1}{4}} \to -\infty$ is

$$V = \tfrac{1}{2}(\epsilon+\epsilon')\,a\Omega + A\exp\,(px/\nu^{\frac{1}{4}}), \tag{4.10}$$

where

$$p^2 = 2\Omega^{\frac{1}{2}}/h. \tag{4.11}$$

For $x > 0$ we similarly have

$$V = B\exp\,(-px/\nu^{\frac{1}{4}}). \tag{4.12}$$

Before proceeding with the solution we remark that: (*a*) the error in (4.9) and consequent error in (4.10) and (4.11) is $O(\nu^{\frac{1}{4}})$; (*b*) these outer shear layers have thickness $\nu^{\frac{1}{4}}h^{\frac{1}{2}}/\Omega^{\frac{1}{4}}$. They can be regarded as thin only if

$$\nu^{\frac{1}{4}}h^{\frac{1}{2}}/\Omega^{\frac{1}{4}} \ll a$$

that is if

$$h/a \ll E^{-\frac{1}{2}}. \tag{4.13}$$

If $h/a \sim E^{-\frac{1}{2}}$ the whole interior of the Taylor column and a comparable exterior region are influenced by viscous forces. We can anticipate that the maximum thickness of the $\frac{1}{3}$ shear layer is $O((\nu h/\Omega)^{\frac{1}{3}})$ and when $h \sim aE^{-\frac{1}{2}}$ this is $O(aE^{\frac{1}{6}})$. Thus there will still be thin $\frac{1}{3}$ layers at the Taylor column boundary if $E^{\frac{1}{6}} \ll 1$ and not until such large separations that $h/a \gg E^{-1}$ will the disks cease to influence each other (see §§ 7 and 8).

The next step in solving the problem is to find A and B. It is clear that no choice of A and B will make V analytic at $x = 0$, so that there must be an inner $\frac{1}{3}$ layer to act as a buffer between the $\frac{1}{4}$ layers. Stewartson (1966) has remarked that the correct boundary conditions on V must be deduced from the properties of the $\frac{1}{3}$ layer and has given a method of determining the correct boundary conditions which we follow closely. We shall find for the present problem, but not in general, contrary to what is sometimes implied in the literature, that continuity of V and $\mathrm{d}V/\mathrm{d}r$ at $x = 0$ is required.

We introduce boundary-layer variables for the $\frac{1}{4}$ and $\frac{1}{3}$ layers ξ, η, where

$$\xi = x/\nu^{\frac{1}{4}} \quad \text{and} \quad \eta = x/\nu^{\frac{1}{3}}, \tag{4.14}$$

so that $\xi = \eta\nu^{\frac{1}{12}}$. If we denote the velocity field in the $\frac{1}{3}$ layer by (u, v, w) we must demand a matching of the layers, expressed symbolically as

$$\lim_{\xi\to\pm 0}(U, V, W) \sim \lim_{\eta\to\pm\infty}(u, v, w). \tag{4.15}$$

If we expand $V(\xi)$ for small ξ we have

$$\left.\begin{aligned} &V \sim \tfrac{1}{2}(\epsilon+\epsilon')\,a\Omega + A(1+p\xi+\tfrac{1}{2}p^2\xi^2+\ldots) \quad \text{as} \quad \xi\to 0- \\ \text{and} \quad &V \sim B(1-p\xi+\tfrac{1}{2}p^2\xi^2+\ldots) \quad \text{as} \quad \xi \to 0+. \end{aligned}\right\} \tag{4.16}$$

$$\left.\begin{aligned} \text{Thus} \quad &v \sim \tfrac{1}{2}(\epsilon+\epsilon')\,a\Omega + A(1+p\eta\nu^{\frac{1}{12}}+\tfrac{1}{2}p^2\eta^2\nu^{\frac{1}{6}}+\ldots) \quad \text{as} \quad \eta\to-\infty \\ \text{and} \quad &v \sim B(1-p\eta\nu^{\frac{1}{12}}+\tfrac{1}{2}p^2\eta^2\nu^{\frac{1}{6}}+\ldots) \quad \text{as} \quad \eta\to+\infty. \end{aligned}\right\} \tag{4.17}$$

This suggests that we write

$$v = v_0(\eta, z) + v_1(\eta, z) + v_2(\eta, z) + \ldots \tag{4.18}$$

and correspondingly

$$w = w_0(\eta, z) + w_1(\eta, z) + w_2(\eta, z) + \ldots \tag{4.19}$$

where for a_n, b_n determined by (17)†

$$\left.\begin{aligned} &v_n \sim a_n\eta^n \quad \text{as} \quad \eta\to-\infty \\ \text{and} \quad &v_n \sim b_n\eta^n \quad \text{as} \quad \eta\to+\infty. \end{aligned}\right\} \tag{4.20}$$

Clearly from the matching v_n and hence w_n for finite η are $O(\nu^{\frac{1}{12}n})$, since it can be anticipated that A and B are of order unity.

† Actually, the error $O(\nu^{\frac{1}{4}})$ in (4.16) will have to be considered when v_3, v_4, ... are being sought. This is because v_3 is $O(\nu^{\frac{1}{4}})$. The difficulty is only one of detail.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Before we can apply Stewartson's method we must complete the specification of the $\frac{1}{3}$ layer problem by finding boundary conditions on $z = 0, h$. Now expressed in terms of η and neglecting the curvature of the coordinate system, the Ekman suction condition becomes

$$w = \pm \frac{1}{2} \frac{\nu^{\frac{1}{6}}}{\Omega^{\frac{1}{2}}} \frac{\partial v}{\partial \eta} \quad \text{on} \quad z = {0 \atop h} \quad (\eta \neq 0), \tag{4.21}$$

so that for $\eta \neq 0$,

$$\left.\begin{aligned} w_0 &= 0 \\ w_1 &= 0 \\ w_2 &= \pm \frac{1}{2} \frac{\nu^{\frac{1}{6}}}{\Omega^{\frac{1}{2}}} \frac{\partial v_0}{\partial \eta} \end{aligned}\right\} \quad \text{on} \quad z = {0 \atop h}. \tag{4.22}$$

However, w_n will in general be singular at $\eta = 0$. The region where the Ekman layer and the $\frac{1}{3}$ layer join can be only $O(\nu^{\frac{1}{2}})$ in breadth so that the $\frac{1}{3}$ layer equations 'see' this join as a singularity (figure 2). The similarity solutions developed in § 3 show that the $\frac{1}{3}$ layer equations permit singularities which are arbitrarily strong, so we cannot limit the strength of the singularity by appeal to the dynamics of the $\frac{1}{3}$ layer. Ideally, we would limit the strength of the singularity by insisting that the $\frac{1}{3}$ layer should fit smoothly onto the joining region—but we have not been able to do this.

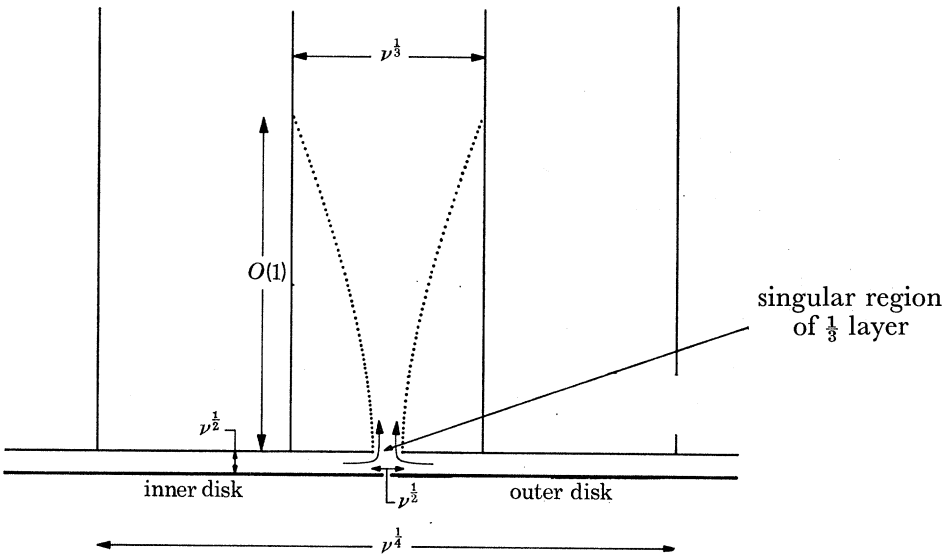


FIGURE 2. Join of the $\frac{1}{3}$ layer to the Ekman layer. The dotted lines enclose the region of more intense shear associated with the singularity at the split.

Instead we impose on the $\frac{1}{3}$ layer solution the extra requirement that its radial pressure gradient be at most of the same order in the large parameter ν^{-1} as the radial pressure gradient in the rest of the flow. This condition removes the non-uniqueness of the $\frac{1}{3}$ layer in a way similar to that in which the Kutta–Joukowsky condition removes the non-uniqueness in the potential flow around a strip—if the condition were not satisfied, there would be an adverse pressure gradient $O(\nu^{-1})$ at the trailing edge. (In the three problems considered in this paper, the 'Kutta condition' is entirely equivalent to an hypothesis of minimum singularity.)

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Since $$-2\Omega v = -\partial p/\partial x,$$

the Kutta condition is equivalent to requiring that the swirl velocity be at most of the same order in ν^{-1} in the $\frac{1}{3}$ layer as in the rest of the flow.

The similarity solutions of § 3 show that

$$v \sim C\eta^{3m} \quad (m = -\tfrac{1}{3}, -\tfrac{2}{3}, \ldots) \tag{4.24}$$

as $\eta/z^{\frac{1}{3}} \to \pm\infty$. Since $m < 0$, v has its largest order of magnitude at the smallest value of η for which (24) is still applicable. Just outside the joining region $x = O(\nu^{\frac{1}{2}})$, so that $\eta = O(\nu^{\frac{1}{6}})$. Now, since $z = O(\nu^{\frac{1}{2}})$, $\eta/z^{\frac{1}{3}}$ is $O(1)$ so that (4.24) will not be a good numerical approximation to v. However, since all the terms in the asymptotic expansion are $O(1)$ it is reasonable to suppose that (4.24) still gives the order of magnitude of v. Thus $v = O(C\nu^{\frac{1}{2}m})$ where the $\frac{1}{3}$ layer meets the Ekman layer. If we apply (4.24) to the terms of (4.18), using the orders of magnitude for a_n, b_n determined by the matching requirement we find $v_n \sim \nu^{\frac{1}{12}(n+6m)}$. The orders of magnitude are listed in the following table:

	$m = -\frac{1}{3}$	$m = -\frac{2}{3}$	$m = -1$
$n = 0$	$\nu^{-\frac{1}{6}}$	$\nu^{-\frac{1}{3}}$	$\nu^{-\frac{1}{2}}$
$n = 1$	$\nu^{-\frac{1}{12}}$	$\nu^{-\frac{1}{4}}$	$\nu^{-\frac{5}{12}}$
$n = 2$	1	$\nu^{-\frac{1}{6}}$	$\nu^{-\frac{1}{3}}$

Thus the Kutta condition requires that the $n = 0$ and $n = 1$ fields be regular at $\eta = 0$ and allows the $n = 2$ field to have at worst an $m = -\frac{1}{3}$ singularity, which corresponds to a δ-function singularity in w. It is worth noting that this means that the $n = 2$ velocity field can draw a flux of amount $\nu^{\frac{1}{2}}$ from the joining region, which is compatible with the flux to the joining region from the Ekman layers on the disks. It is sometimes asserted that the $\frac{1}{3}$ layer occurs because of the need to transport fluid from one Ekman layer to the other. This is misleading and will lead in general to errors if the orders of magnitude in the $\frac{1}{3}$ layer are fixed in this way. In fact, we shall find that in general the flow in the $\frac{1}{3}$ layer is *recirculating* to leading order in ν^{-1}, and is of a larger order than flux requirements would suggest.

We can now apply Stewartson's method to determine jump conditions. We have

$$\frac{\partial^3 v_0}{\partial \eta^3} = -2\Omega \frac{\partial w_0}{\partial z}, \tag{4.25}$$

so that $$\frac{\partial^3}{\partial \eta^3}\int_0^h v_0 \, dz = -2\Omega[w_0(\eta, h) - w_0(\eta, 0)]. \tag{4.26}$$

Our discussion of the singularities of the $n = 0$ velocity field coupled with the boundary conditions (4.22) show that

$$w_0(\eta, h) = w_0(\eta, 0) = 0 \quad \text{all} \quad \eta. \tag{4.27}$$

Hence $$\int_0^h v_0 \, dz = d_0^{(0)} + d_1^{(0)}\eta + d_2^{(0)}\eta^2. \tag{4.28}$$

But v_0 is independent of z as $\eta \to \pm\infty$, since v_0 must match to the $\frac{1}{4}$ layer swirl velocity, so that (4.20) gives $d_1^{(0)} = d_2^{(0)} = 0$ and

$$hv_0 \to d_0^{(0)} \quad \text{as} \quad \eta \to \pm\infty. \tag{4.29}$$

Thus
$$\tfrac{1}{2}(\epsilon+\epsilon')\,a\Omega+A=B \tag{4.30}$$
which is the first required jump condition. It is equivalent to requiring that V be continuous at $\xi=0$.

A similar argument applied to v_1 gives
$$Ap\nu^{\frac{1}{12}}=-Bp\nu^{\frac{1}{12}} \tag{4.31}$$
which gives the second jump condition, equivalent to requiring continuity of $\mathrm{d}V/\mathrm{d}r$ at $\xi=0$.

The $\frac{1}{4}$ layers are now completely determined and
$$\left.\begin{aligned}V&=\tfrac{1}{2}(\epsilon+\epsilon')\,a\Omega\{1-\tfrac{1}{2}\,\mathrm{e}^{p\xi}\}\quad(\xi<0),\\ V&=\tfrac{1}{4}(\epsilon+\epsilon')\,a\Omega\,\mathrm{e}^{-p\xi}\quad(\xi>0).\end{aligned}\right\} \tag{4.32}$$
In the special case $\epsilon+\epsilon'=0$, in which the disks have equal and opposite rotation, the $\frac{1}{4}$ layer disappears. In this case the core of the Taylor column has the same angular velocity as the exterior rigid rotation, so that the $\frac{1}{4}$ layer in this problem is associated with a discontinuity in the geostrophic swirl.

We can easily find W and the result is
$$W=-{}^{1}(\epsilon+\epsilon')\,\nu^{\frac{1}{4}}\,\Omega^{\frac{1}{2}}pa\left(1-\frac{2z}{h}\right)\mathrm{e}^{-p|\xi|}, \tag{4.33}$$
so that there is a non-constant flux
$$-\tfrac{1}{2}\pi a^2(\epsilon+\epsilon')\,(\nu\Omega)^{\frac{1}{2}}\left(1-\frac{2z}{h}\right) \tag{4.34}$$
in the $\frac{1}{4}$ layers.

It remains to determine the structure of the $\frac{1}{3}$ layer. Clearly $v_0=d_0^{(0)}$, $w_0=0$ and $v_1=d_1^{(1)}\eta$, $w_1=0$. Thus the $n=2$ velocity field gives the structure of the $\frac{1}{3}$ layer.

It is determined by the boundary conditions
$$w_2=C'\,\delta(\eta)\quad\text{on}\quad z=0,\qquad w_2=C\,\delta(\eta)\quad\text{on}\quad z=h \tag{4.35}$$
and
$$v_2\sim\pm b\eta^2\quad\text{as}\quad\eta\to\pm\infty, \tag{4.36}$$
where in view of (17)
$$b=\tfrac{1}{8}(\epsilon+\epsilon')\,p^2\nu^{\frac{1}{6}}. \tag{4.37}$$
The flux into the $\frac{1}{3}$ layer from the Ekman layer on $z=0$ is $2\pi a\nu^{\frac{1}{3}}C'$ and the flux into the $\frac{1}{3}$ layer from the Ekman layer on $z=h$ is $-2\pi a\nu^{\frac{1}{3}}C$. Now in view of (4.5) and (4.34) the Ekman layer on $z=0$ is receiving a net flux $\pi a^2\epsilon'(\Omega\nu)^{\frac{1}{2}}$ from the geostrophic flow and the $\frac{1}{4}$ layers, while the Ekman layer on $z=h$ is receiving a net flux $\pi a^2\epsilon(\nu\Omega)^{\frac{1}{2}}$. Thus by continuity
$$C'=\tfrac{1}{2}\nu^{\frac{1}{6}}\Omega^{\frac{1}{2}}\epsilon' a\quad\text{and}\quad C=-\tfrac{1}{2}\nu^{\frac{1}{6}}\Omega^{\frac{1}{2}}\epsilon a. \tag{4.38}$$
(It is perhaps worth mentioning that (4.35) and (4.38) can be derived formally from the Ekman suction condition. Thus on $z=0$,
$$w=\frac{1}{2}\frac{\nu^{\frac{1}{6}}}{\Omega^{\frac{1}{2}}}\frac{\partial v}{\partial\eta}-\tfrac{1}{2}\epsilon'(\nu\Omega)^{\frac{1}{2}}\frac{\partial}{\partial r}[rH(a-r)],$$
and hence on differentiating the Heaviside function, replacing r by η, and equating coefficients of $\nu^{\frac{1}{6}}$, we have
$$w_2=+\tfrac{1}{2}\epsilon' a\nu^{\frac{1}{6}}\Omega^{\frac{1}{2}}\,\delta(\eta)$$
and similarly for $z=h$.)

Let $\chi = w_2 + \mathrm{i}v_2$. Then

$$\frac{\partial^3 \chi}{\partial \eta^3} = -2\Omega \mathrm{i} \frac{\partial \chi}{\partial z}. \tag{4.39}$$

We define the Fourier transform by

$$\tilde{\chi} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\alpha\eta} \chi(\eta, z)\, \mathrm{d}\eta \tag{4.40}$$

and on taking the transform of (4.39) we get

$$\frac{\partial \tilde{\chi}}{\partial z} = -\frac{\alpha^3}{2\Omega} \tilde{\chi} \tag{4.41}$$

so that

$$\tilde{\chi}(\alpha, z) = A(\alpha)\, \mathrm{e}^{-\alpha^3 z/2\Omega}. \tag{4.42}$$

It is clear from the boundary conditions that w_2 is an even function and v_2 is an odd function of η. Thus $\tilde{\chi}$ is real and so

$$w_2(\eta, z) = \int_0^{\infty} \cos \alpha x (\tilde{\chi}(\alpha, z) + \tilde{\chi}(-\alpha, z))\, \mathrm{d}\alpha. \tag{4.43}$$

The boundary conditions on w_2 at once give

$$\left.\begin{aligned} &A(\alpha) + A(-\alpha) = C'/\pi \quad (\alpha > 0) \\ \text{and}\quad &A(\alpha)\, \mathrm{e}^{-\alpha^3 h/2\Omega} + A(-\alpha)\, \mathrm{e}^{\alpha^3 h/2\Omega} = C/\pi \quad (\alpha > 0), \end{aligned}\right\} \tag{4.44}$$

$$\left.\begin{aligned} \text{so that}\quad A(\alpha) &= \frac{C'\, \mathrm{e}^{\alpha^3 h/2\Omega} - C}{2\pi \sinh(\alpha^3 h/2\Omega)} \quad (\alpha > 0) \\ \text{and}\quad A(-\alpha) &= \frac{C - C'\, \mathrm{e}^{-\alpha^3 h/2\Omega}}{2\pi \sinh(\alpha^3 h/2\Omega)} \quad (\alpha > 0). \end{aligned}\right\} \tag{4.45}$$

The conditions (4.36) require that $A(\alpha)$ be singular at $\alpha = 0$ and, precisely,

$$A(\alpha) \sim 2b/\pi\alpha^3 \quad \text{as} \quad \alpha \to 0, \tag{4.47}$$

so that from (4.45)

$$C' - C = 2bh/\Omega. \tag{4.48}$$

We can easily verify that the values of C' and C given by (4.38) satisfy this requirement. This is not fortuitous, but a consequence of the identity

$$h\left(\frac{\partial^2 v}{\partial \eta^2}\right)_{+\infty} - h\left(\frac{\partial^2 v}{\partial \eta^2}\right)_{-\infty} = -2\Omega \int_{-\infty}^{\infty} [w(\eta, h) - w(\eta, 0)]\, \mathrm{d}\eta \tag{4.49}$$

satisfied by solutions of the $\frac{1}{3}$ layer equations.

Thus the $\frac{1}{3}$ layer velocity field is given by

$$w_2 + \mathrm{i}v_2 = \tfrac{1}{2}\nu^{\frac{1}{6}}\Omega^{\frac{1}{2}} a \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\alpha\eta} \frac{\mathrm{e}^{-\alpha^3 z/2\Omega}(\epsilon'\, \mathrm{e}^{\alpha^3 h/2\Omega} + \epsilon)}{2\pi \sinh(\alpha^3 h/2\Omega)}\, \mathrm{d}\alpha \tag{4.50}$$

and it is a straightforward matter to evaluate the integral in (4.50) be residues to recover Stewartson's (1957) results.

One feature of the solution is worth mentioning. The contribution, $\chi(\epsilon')$ say, to $w_2 + \mathrm{i}v_2$ from the term in (4.50) involving ϵ' can be written

$$\chi(\epsilon') = \frac{1}{2} \frac{\nu^{\frac{1}{6}}\Omega^{\frac{1}{2}}\epsilon' a}{\pi} \left\{ \int_0^{\infty} \mathrm{e}^{-\mathrm{i}\alpha\eta} \frac{\mathrm{e}^{-\alpha^3 z/2\Omega}}{1 - \mathrm{e}^{-\alpha^3 h/\Omega}}\, \mathrm{d}\alpha - \int_0^{\infty} \mathrm{e}^{\mathrm{i}\alpha\eta} \frac{\mathrm{e}^{(\frac{1}{2}\alpha^3 z - \alpha^3 h)/\Omega}}{1 - \mathrm{e}^{-\alpha^3 h/\Omega}}\, \mathrm{d}\alpha \right\},$$

so that on expanding the denominators, we get

$$\chi(\epsilon') = \frac{1}{2}\frac{\nu^{\frac{1}{6}}\Omega^{\frac{1}{2}}\epsilon' a}{\pi}\left\{\int_0^\infty \mathrm{e}^{-\mathrm{i}\alpha\eta}\left(\sum_{n=0}^{\infty}\mathrm{e}^{-(\alpha^3 z-2n\alpha^3 h)/2\Omega}\right)\mathrm{d}\alpha - \int_0^\infty \mathrm{e}^{\mathrm{i}\alpha\eta}\left(\sum_{n=1}^{\infty}\mathrm{e}^{(\alpha^3 z-2n\alpha^3 h)/2\Omega}\right)\mathrm{d}\alpha\right\}.$$

If we use the form given in § 3 for the flow field due to a line source, we can write this as

$$\chi(\epsilon') = \frac{(2\Omega)^{\frac{1}{3}}}{2}\frac{\nu^{\frac{1}{6}}\Omega^{\frac{1}{2}}\epsilon' a}{\pi}\sum_{n=0}^{\infty}R(\eta, z+2nh) - \frac{(2\Omega)^{\frac{1}{3}}}{2}\frac{\nu^{\frac{1}{6}}\Omega^{\frac{1}{2}}\epsilon' a}{\pi}\sum_{n=1}^{\infty}R^*(\eta, -z+2nh). \quad (4.51)$$

According to the definition of R—see (3.25) and (3.26)—$\chi(\epsilon')$ is due to a line source of strength $\nu^{\frac{1}{2}}\Omega^{\frac{1}{2}}\epsilon' a$ per unit length at $\eta = 0$, $z = 0$ (which gives an upward flux of one-half this amount) and equal image sources at $\eta = 0$, $z = \pm 2nh$. $\chi(\epsilon)$ can be displayed similarly. This representation brings out clearly the singular nature of the $\frac{1}{3}$ layer at the plates. Moreover, it makes it clear that the discontinuity is responsible for the singularity since, from (51), the singularity at $\eta = 0$, $z = 0$ disappears when $\epsilon' = 0$.

5. The rising disk

The plane $z = -h_B$ is rigid and rotates about Oz with angular velocity Ω. The plane $z = h_T$ is either a second rigid plane rotating about Oz with angular velocity Ω or it is a free surface ($g/a\Omega^2$ is assumed to be so large that curvature effects are negligible). The rigid disk $z = 0$, $0 \leqslant r \leqslant a$ is rising with a prescribed velocity U ($\ll a\Omega$) parallel to Oz and has a prescribed angular velocity $\Omega(1+\epsilon)$ about Oz.

The geostrophic motion and the resulting forces on the disk were determined by Moore & Saffman (1968). In particular, for the case of a disk rising under its own buoyancy the condition that the torque be zero determines ϵ. When the upper surface is rigid $\epsilon = 0$, but when it is free it follows from their analysis that

$$\epsilon = -UE^{-\frac{1}{2}}/a. \quad (5.1)$$

We shall not impose this condition on ϵ in the present calculation, but we must bear in mind that ϵ can be as large as $\nu^{-\frac{1}{2}}$, a point which is important when the ordering is being carried out.

The geostrophic motion can be determined by considering the flux in the Ekman layers and, for $h_T > z > 0$,

$$-\tfrac{1}{2}Q\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}V_G - \frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}(V_G - \epsilon\Omega r) = \tfrac{1}{2}Ur \quad (0 \leqslant r < a), \quad (5.2)$$

since $U_G = 0$, where $Q = 0$ if the upper surface is free and $Q = 1$ if it is rigid. Thus

$$V_{GT}(r) = \frac{-r(\Omega/\nu)^{\frac{1}{2}}\{U - \epsilon\Omega(\nu/\Omega)^{\frac{1}{2}}\}}{1+Q} \quad (0 \leqslant r < a), \quad (5.3)$$

while for $-h_B < z < 0$

$$V_{GB}(r) = \tfrac{1}{2}r\,(\Omega/\nu)^{\frac{1}{2}}\{U + \epsilon\Omega(\nu/\Omega)^{\frac{1}{2}}\} \quad (0 \leqslant r < a). \quad (5.4)$$

Note that the swirl velocity of the geostrophic flow is now $O(\nu^{-\frac{1}{2}})$. Finally, for $r > a$ and $-h_B < z < h_T$

$$V_G = 0. \quad (5.5)$$

We remark that when $Q = 1$ and $\epsilon = 0$ the geostrophic swirl is antisymmetric even when $h_T + h_B$. However, the $\frac{1}{4}$ and $\frac{1}{3}$ layers will *not* in general possess this symmetry, which is purely a property of the geostrophic flow.

The $\frac{1}{4}$ layers are obtained just as in §4 and we find

$$V_T(\xi) = V_{GT}(a) + A_T e^{p_T \xi} \quad (z > 0,\ \xi < 0), \tag{5.6}$$

$$V_B(\xi) = V_{GB}(a) + A_B e^{p_B \xi} \quad (z < 0,\ \xi < 0) \tag{5.7}$$

and

$$V(\xi) = B\, e^{-p\xi} \quad (\xi > 0), \tag{5.8}$$

where p_T, p_B and p are defined by (5.28) in terms of h_T, h_B and h. Once again, the difficult part of the calculation is the determination from the $\frac{1}{3}$ layer equations of jump conditions. If we expand the $\frac{1}{4}$ layer solutions about $\xi = 0$ and replace ξ by η in the expansions we find that the $\frac{1}{3}$ layer velocity field can be decomposed into components just as in (4.18), (4.19) and (4.20) and we now follow that analysis closely. However, $v_n(\eta, z)$ and $w_n(\eta, z)$ are now $O(\nu^{\frac{1}{12}(n-6)})$.

We now must examine the possible singularities of the velocity field v_n at $\eta = 0$ and $z = 0$. The similarity solutions of §3 show that in addition to the type discussed in §4 a new series is possible in which

$$v \sim \eta^{3m} \quad (m = -\tfrac{1}{6}, -\tfrac{1}{2}, \ldots).$$

The fact that the geometry of the boundaries near the singularity differs from that of the split disk is responsible for this new series of values of m. In particular, the weakest possible singularity is less strong than the weakest singularity for the split disk, and this permits a stronger $\frac{1}{3}$ layer to appear.

An argument like that of §4 gives the maximum orders of magnitude of v_n, shown in the table below, in the region where the $\frac{1}{3}$ layer meets the Ekman layer:

	$m = -\frac{1}{6}$	$m = -\frac{1}{3}$	$m = -\frac{1}{2}$
$n = 0$	$\nu^{-\frac{7}{12}}$	$\nu^{-\frac{8}{12}}$	$\nu^{-\frac{9}{12}}$
$n = 1$	$\nu^{-\frac{1}{2}}$	$\nu^{-\frac{7}{12}}$	$\nu^{-\frac{8}{12}}$
$n = 2$	$\nu^{-\frac{5}{12}}$	$\nu^{-\frac{1}{2}}$	$\nu^{-\frac{7}{12}}$

Now the swirl velocity in the region exterior to the shear layers is $O(\nu^{-\frac{1}{2}})$ so that the Kutta condition requires that (*a*) v_0 and w_0 be regular at $\eta = 0, z = 0$; (*b*) v_1 and w_1 have at most an $m = -\frac{1}{6}$ singularity corresponding to singularities of v_1 and w_1 on $z = 0$ like $|\eta|^{-\frac{1}{2}}$; (*c*) v_2 and w_2 can have in addition to the $m = -\frac{1}{6}$ singularity an $m = -\frac{1}{3}$ singularity, corresponding to a δ-function behaviour of w_2.

The boundary conditions on v_n, w_n follow from the Ekman suction relations. These are for rigid surfaces

$$w = \mp \frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \frac{1}{r}\frac{d}{dr}(rv) \quad \text{on} \quad z = \begin{matrix} h_T \\ -h_B \end{matrix} \quad (\text{all } r), \tag{5.9}$$

$$w - U = \pm \frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \frac{1}{r}\frac{d}{dr}(rv - \epsilon\Omega r^2) \quad \text{on} \quad z = 0\pm \quad (r < a). \tag{5.10}$$

If the upper surface is free, the condition there is $w = 0$. Changing to the boundary layer variable η, inserting the expansion (4.18) and (4.19) and the dependence on ν required by the matching with the $\frac{1}{4}$-layer solutions ((5.6) to (5.8)) we find (see (4.22))

$$\left.\begin{aligned} w_0 = w_1 = 0 \quad \text{on} \quad & z = h_T, -h_B \quad (-\infty < \eta < \infty), \\ \text{and} \qquad & z = 0\pm \quad (\eta < 0), \end{aligned}\right\} \tag{5.11}$$

and
$$w_2 = \pm \frac{Q\nu^{\frac{1}{6}}}{2\Omega^{\frac{1}{2}}} \frac{\partial v_0}{\partial \eta} \tag{5.12}$$

on the appropriate surfaces, where $Q = 1$ unless the upper surface is free where it is then zero. Note that to this order the velocity of rise and the relative rotation of the disk do not enter the boundary conditions for the $\frac{1}{3}$ layer velocity field, which is therefore determined to this order by its matching with the $\frac{1}{4}$ layers. Since the error in the $\frac{1}{3}$ layer equations is $O(\nu^{\frac{1}{3}})$ (see §2), the pairs w_n, v_n for $n = 0, 1, 2$, satisfy

$$\frac{\partial^3 v_n}{\partial \eta^3} = -2\Omega \frac{\partial w_n}{\partial z}, \quad \frac{\partial^3 w_n}{\partial \eta^3} = 2\Omega \frac{\partial v_n}{\partial z}. \tag{5.13}$$

We can now derive some jump conditions across the central layer. Integrating the first equation of (5.13) from 0 to h_T and $-h_B$ to 0, we obtain

$$\frac{\partial^3}{\partial \eta^3} \int_0^{h_T} v_0 \, dz = 2\Omega w_0(\eta, 0+) \quad \text{and} \quad \frac{\partial^3}{\partial \eta^3} \int_{-h_B}^{0} v_0 \, dz = -2\Omega w_0(\eta, 0-). \tag{5.14}$$

Now $w_0(\eta, 0)$ vanishes for $\eta < 0$ and is continuous across $z = 0$ for $\eta > 0$. Hence,

$$\frac{\partial^3}{\partial \eta^3} \int_{-h}^{h_T} v_0 \, dz = 0, \tag{5.15}$$

so that on integrating we have

$$\int_{-h_B}^{h_T} v_0 \, dz = d_0^{(0)} + d_1^{(0)} \eta + d_2^{(0)} \eta^2. \tag{5.16}$$

By definition v_0 is bounded at $\pm\infty$ (cf. (4.20)) so that $d_1^{(0)} = d_2^{(0)} = 0$ and we deduce that

$$h_T(V_{GT}(0) + A_T) + h_B(V_{GB}(0) + A_B) = hB \tag{5.17}$$

or equivalently
$$h_T V_T(0) + h_B V_B(0) = hV(0) \tag{5.18}$$

so that we have not been able to prove continuity of V by the present argument.

We can apply the same argument to v_1 and w_1. The singularity of w_1 at $\eta = 0$, $z = 0$ occasions no difficulty because the $m = -\frac{1}{6}$ singularity gives a w_1 continuous across $z = 0$. Proceeding as before, we find

$$\frac{\partial^3}{\partial \eta^3} \int_{-h_B}^{h_T} v_1 \, dz = 0, \tag{5.19}$$

from which we deduce

$$\int_{-h_B}^{h_T} v_1 \, dz = d_0^{(1)} + d_1^{(1)} \eta + d_2^{(1)} \eta^2. \tag{5.20}$$

Matching with the conditions at $\eta = \pm\infty$, we find $d_0^{(1)} = d_2^{(1)} = 0$ and

$$h_T A_T p_T + h_B A_B p_B = -hBp, \tag{5.21}$$

or equivalently
$$h_T V_T'(0) + h_B V_B'(0) = hV'(0). \tag{5.22}$$

The condition (5.22) means that the total tangential stress is continuous across the inner layer.

A physical argument for (5.22) is as follows. The net flux of angular momentum from the $\frac{1}{4}$ layers into the $\frac{1}{3}$ layer by the azimuthal viscous stresses across the vertical sides (per unit length of circumference and per unit mass) is

$$\int_0^{h_T} \nu \frac{\partial v}{\partial r} dz + \int_{-h_B}^{0} \nu \frac{\partial v}{\partial r} dz - \int_{-h_B}^{h_T} \nu \frac{\partial v}{\partial r} dz. \tag{5.23}$$

Using the orders of magnitude for the $\frac{1}{4}$ layer which are appropriate at the central edges, we obtain that (5.23) has order of magnitude $\nu h U(\Omega a^2/\nu)^{\frac{1}{2}}/\delta_{\frac{1}{4}} \sim \nu^{\frac{1}{4}}$. This angular momentum flux balances the net flux of angular momentum convected into the $\frac{1}{3}$ layer by the total radial flux (this is proportional to the total mass flux into the $\frac{1}{3}$ layer multiplied by the change in the absolute rotation velocity $(\Omega\delta_{\frac{1}{3}})$ and is of order $Uh\Omega\delta_{\frac{1}{3}} \sim \nu^{\frac{1}{3}}$) plus the torque on the parts of the disk and end walls covered by the $\frac{1}{3}$ layer (this is proportional to the stress across the Ekman layer times $\delta_{\frac{1}{3}}$ and is of order $U\Omega a\delta_{\frac{1}{3}} \sim \nu^{\frac{1}{3}}$). There are also terms proportional to the squares of the perturbation velocities but these are ignored on the assumption of zero Rossby number. From the dependence with ν, it is clear that the conservation of angular momentum requires that (5.23) vanish, from which follows (5.22). (We have not discovered a physical explanation of condition (5.18) on the velocities, but as will be seen below it appears that the Kutta condition imposes restrictions which enable (5.18) to be satisfied identically, so that it is in a sense superfluous.)

The two relations so far obtained between the three constants A_T, A_B, B are not sufficient and a third independent relation is required. We can apply the above argument to v_2 and w_2. It is more difficult because of the more complicated Ekman condition (5.12), and because the δ-function or source singularity of w_2 at the edge must be taken into account. We do not give the details, because the end result (see (5.30) below) is essentially an equation for the unknown source strength and is a statement that mass flux is conserved in the $\frac{1}{3}$ layer and thereby gives no further information.

It is therefore necessary to argue differently and use the results of the Kutta condition, according to which v_0 and w_0 are non-singular. A solution of the $\frac{1}{3}$ layer equations, which is free of singularities, has bounded v_0 as $\eta \to \pm\infty$, and satisfies the boundary conditions on w_0, is $v_0 =$ constant, $w_0 = 0$. It seems plausible (and we will shortly prove conclusively) that this is the only solution with these properties. Clearly if this is so, we must have

$$V_T(0) = V_B(0) = V(0). \tag{5.24}$$

The jump condition (5.18) is now redundant and we are left with the required three conditions to determine the $\frac{1}{4}$ layers.

It is perhaps worth stressing that V' is not continuous across the $\frac{1}{3}$ layer. The condition (5.22) is a weaker condition on V' which implies merely that the net tangential viscous stress is continuous across the $\frac{1}{3}$ layer.

We can now easily find the constants A_T, A_B and B and we have

$$A_T = B + \frac{a\Omega^{\frac{1}{2}}}{\nu^{\frac{1}{2}}}\left[U - \epsilon\Omega\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\right]\Big/(1+Q) \tag{5.25}$$

and

$$A_B = B - \frac{a\Omega^{\frac{1}{2}}}{\nu^{\frac{1}{2}}}\left[U + \epsilon\Omega\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\right]\Big/2 \tag{5.26}$$

where

$$B = \frac{a\left(\dfrac{\Omega}{\nu}\right)^{\frac{1}{2}}\left\{\dfrac{h_T p_T}{1+Q}\left[-U + \epsilon\Omega\left(\dfrac{\nu}{\Omega}\right)^{\frac{1}{2}}\right] + \frac{1}{2}h_B p_B\left[U + \epsilon\Omega\left(\dfrac{\nu}{\Omega}\right)^{\frac{1}{2}}\right]\right\}}{(h_T p_T + h_B p_B + hp)}, \tag{5.27}$$

and where

$$\left.\begin{aligned} p_T^2 &= \Omega^{\frac{1}{2}}(1+Q)/h_T, \\ p_B^2 &= 2\Omega^{\frac{1}{2}}/h_B \\ \text{and} \quad p^2 &= \Omega^{\frac{1}{2}}(1+Q)/h. \end{aligned}\right\} \tag{5.28}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Two features of this solution are worth noting. (i) For the disk rising freely between a pair of rigid planes $Q = 1$ and $\epsilon = 0$. Then (5.27) and (5.28) show that the external $\frac{1}{4}$ layer is absent when $h_T = h_B$, that is to say, when the disk is midway between the planes. (ii) Both Ekman layers on the disk have an outward radial flux $\frac{1}{2}M$ per unit length of circumference where

$$M = -(\nu/\Omega)^{\frac{1}{2}}[V(0) - \epsilon\Omega a]. \tag{5.29}$$

Thus there is a net flux into the $n = 2$ component of the $\frac{1}{3}$ layer (as we have seen, this carries $O(1)$ flux) which comes out as a source singularity strength M in the $\frac{1}{4}$ layer solution with $n = 2$ (figure 3). The argument leading to (5.17) and (5.21) applied to v_2, w_2 gives

$$hV''(0) - h_T V_T''(0) - h_B V_B''(0) = -2\Omega^{\frac{1}{2}}[V(0) - \epsilon\Omega a], \tag{5.30}$$

which is easily seen to be satisfied by our solution for the $\frac{1}{4}$ layers.

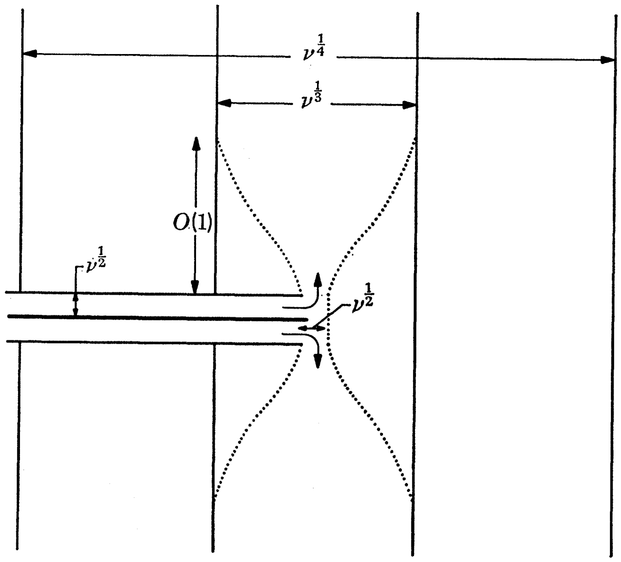


FIGURE 3. Join of the $\frac{1}{3}$ layer to the Ekman layer in the disk case. Dotted line encloses the equal and opposite fluxes from the Ekman layer which give the $n = 2$ field.

It remains to verify our conjecture about the $n = 0$ component of the $\frac{1}{3}$ layer and to determine the structure of the $n = 1$ and $n = 2$ components. For the $n = 0, 1$ fields we must consider the problem of solving†

$$\frac{\partial^6 v_n}{\partial \eta^6} = -4\Omega^2 \frac{\partial^2 v_n}{\partial z^2} \quad (n = 0, 1), \tag{5.31}$$

with boundary conditions

$$\frac{\partial v_n}{\partial z} = 0 \quad \text{on} \quad z = -h_B \quad (\text{all } \eta \quad n = 0, 1), \tag{5.32}$$

$$\frac{\partial v_n}{\partial z} = 0 \quad \text{on} \quad z = h_T \quad (\text{all } \eta \quad n = 0, 1). \tag{5.33}$$

and

$$\frac{\partial v_n}{\partial z} = 0 \quad \text{on} \quad z = 0 \quad (-\infty < \eta < 0 \quad n = 0, 1). \tag{5.34}$$

† Since there are no symmetries about $\eta = 0$, there is no advantage in using χ as $\tilde{\chi}$ will be no longer real.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The boundary conditions at $\pm\infty$ are that,

$$v_n \to d_B^{(n)}\eta^n \quad (\eta\to-\infty \quad -h_B \leqslant z \leqslant 0), \tag{5.35}$$

$$v_n \to d_T^{(n)}\eta^n \quad (\eta\to-\infty \quad 0 \leqslant z \leqslant h_T) \tag{5.36}$$

and

$$v_n \to d^{(n)}\eta^n \quad (\eta\to+\infty \quad -h_B \leqslant z \leqslant h_T), \tag{5.37}$$

where $d^{(n)} = (-1)^n (\nu^{\frac{1}{12}n}/n!)\, d^nV/\mathrm{d}\xi^n$ evaluated at $\xi = 0+$, etc. To avoid trouble with the Fourier transform we write

$$v_n = d_B^{(n)}\eta^n + v \quad (-\infty \leqslant \eta \leqslant 0, \quad -h_B \leqslant z \leqslant 0), \tag{5.38}$$

etc. Thus we require

$$v \to 0 \quad (\eta\to\pm\infty) \tag{5.39}$$

and that v satisfies (5.32), (5.33) and (5.34).

Defining the Fourier transform as in (4.40), we find, allowing for the singularity of v at $\eta = 0$, that

$$\frac{\mathrm{d}^2\tilde{v}}{\mathrm{d}z^2} - \frac{\alpha^6\tilde{v}}{4\Omega^2} - \frac{\alpha^{5-n}}{4\Omega^2}J^{(n)} = 0 \tag{5.40}$$

where

$$J^{(n)} = \begin{cases} J_T^{(n)} = (2\pi)^{-1}\, n!\, i^{n+1}[d^{(n)} - d_T^{(n)}] & (z>0) \\ J_B^{(n)} = (2\pi)^{-1}\, n!\, i^{n+1}[d^{(n)} - d_B^{(n)}] & (z<0), \end{cases} \tag{5.41}$$

which has the general solutions

$$\begin{aligned} \tilde{v}_T &= a_T(\alpha)\, \mathrm{e}^{\alpha^3 z/2\Omega} + b_T(\alpha)\, \mathrm{e}^{-\alpha^3 z/2\Omega} + J_T^{(n)}/\alpha^{n+1} \\ \tilde{v}_B &= a_B(\alpha)\, \mathrm{e}^{\alpha^3 z/2\Omega} + b_B(\alpha)\, \mathrm{e}^{-\alpha^3 z/2\Omega} + J_B^{(n)}/\alpha^{n+1}. \end{aligned} \tag{5.42}$$

The boundary conditions (5.32) and (5.33) gives

$$b_T(\alpha) = a_T(\alpha)\, \mathrm{e}^{\alpha^3 h_T/\Omega} \quad \text{and} \quad b_B(\alpha) = a_B(\alpha)\, \mathrm{e}^{-\alpha^3 h_B/\Omega}. \tag{5.43}$$

Now $\partial v/\partial z$ is continuous across $z = 0$ for $\eta > 0$ while (5.34) implies that it is continuous across $z = 0$ for $\eta < 0$. Thus $\partial\tilde{v}/\partial z$ is continuous across $z = 0$ and so

$$\alpha^3 a_T(\alpha)\,(1 - \mathrm{e}^{\alpha^3 h_T/\Omega}) = \alpha^3 a_B(\alpha)\,(1 - \mathrm{e}^{-\alpha^3 h_B/\Omega}). \tag{5.44}$$

We define

$$\tilde{v}_+(\alpha, z) = \frac{1}{2\pi}\int_0^\infty \mathrm{e}^{\mathrm{i}\alpha\eta}\, v(\eta, z)\,\mathrm{d}\eta \quad \text{and} \quad \tilde{v}_-(\alpha, z) = \frac{1}{2\pi}\int_{-\infty}^0 \mathrm{e}^{\mathrm{i}\alpha\eta}\, v(\eta, z)\,\mathrm{d}\eta \tag{5.45}$$

and introduce the condensed notation

$$\lim_{z\to 0+} \tilde{v}_+(\alpha, z) = \tilde{v}_+(0+) \quad \text{etc.} \tag{5.46}$$

We do not know the discontinuity of v across $z = 0$, $\eta < 0$ and we do not know the value of $\partial v/\partial z$ on $z = 0$, $\eta > 0$. But only one of our original four functions a_T, a_B, b_T, b_B is still unknown. Thus there must be a relation between the transform $\tilde{v}_-(0+) - \tilde{v}_-(0-)$ of the unknown discontinuity and the transform $(\partial\tilde{v}_+(0)/\partial z)$ of the unknown derivative. This relation turns out to be

$$\tilde{v}_-(0+) - \tilde{v}_-(0-) - \frac{Z^{(n)}}{\alpha^{n+1}} = -\frac{\partial\tilde{v}_+(0)}{\partial z}\,\frac{2\Omega \sinh[\alpha^3(h_T+h_B)/2\Omega]}{\alpha^3 \sinh(\alpha^3 h_T/2\Omega)\sinh(\alpha^3 h_B/2\Omega)}, \tag{5.47}$$

where

$$Z^{(n)} = J_T^{(n)} - J_B^{(n)}. \tag{5.48}$$

Now it is shown in the Appendix that we can write

$$\frac{2\Omega \sinh\left[\alpha^3(h_T+h_B)/2\Omega\right]}{\alpha^3 \sinh(\alpha^3 h_T/2\Omega)\sinh(\alpha^3 h_B/2\Omega)} = \frac{1}{\alpha^6}\frac{S_+(\alpha)}{S_-(\alpha)} \tag{5.49}$$

where the functions S_+, S_- have the properties that

$$\left.\begin{array}{lll} S_+(\alpha) & \text{is analytic and non-zero in} & \frac{7}{6}\pi > \arg\alpha > -\frac{1}{6}\pi, \\ S_-(\alpha) & \text{is analytic and non-zero in} & -\frac{7}{6}\pi < \arg\alpha < -\frac{1}{6}\pi, \end{array}\right\} \tag{5.50}$$

they are bounded and equal to one at $\alpha = 0$, and

$$\left.\begin{array}{lll} S_+(\alpha) \sim s_+\alpha^{\frac{3}{2}} & \text{as} & \alpha \to \mathrm{i}\infty \\ S_-(\alpha) \sim s_-\alpha^{-\frac{3}{2}} & \text{as} & \alpha \to -\mathrm{i}\infty. \end{array}\right\} \tag{5.51}$$

Thus we can write (5.47) as

$$\alpha^6 S_-(\alpha)\left[\tilde{v}_-(0+)-\tilde{v}_-(0-)-\frac{Z^{(n)}}{\alpha^{n+1}}\right] = -\frac{\partial\tilde{v}_+(0)}{\partial z}S_+(\alpha). \tag{5.52}$$

The left-hand side is analytic in some lower half-plane and the right-hand side is analytic in some upper half-plane. Assuming that these half-planes overlap—and by (5.50) this will be the case if the half-planes of analyticity of $\partial\tilde{v}_+/\partial z$ and $\tilde{v}_-(0+)-\tilde{v}_-(0-)$ overlap—each side of (5.52) is the analytic continuation of some function analytic in the entire α plane.† The next step is to determine this entire function by using Liouville's theorem.

Suppose that

$$v(\eta, 0+)-v(\eta, 0-) \sim |\eta|^{-p} \quad \text{as} \quad \eta \to 0-. \tag{5.53}$$

Then w will have the same singularity as $\eta \to 0+$ and so

$$\frac{\partial v}{\partial z} \sim \frac{\partial^3 w}{\partial \eta^3} \sim \eta^{-p-3} \quad \text{as} \quad \eta \to 0+ \tag{5.54}$$

As a consequence of these 'edge' conditions

$$\left.\begin{array}{lll} \tilde{v}_-(0+)-\tilde{v}_-(0-) \sim \alpha^{p-1} & \text{as} & \alpha \to -\mathrm{i}\infty \\ \text{and} \quad \partial\tilde{v}_+(0)/\partial z \sim \alpha^{p+2} & \text{as} & \alpha \to \mathrm{i}\infty. \end{array}\right\} \tag{5.55}$$

Thus both sides of (5.52) are $O(\alpha^{p+\frac{7}{2}})$ for large α and so the entire function in question must be

$$E(\alpha) = A_0 + A_1\alpha + \ldots + A_{p+\frac{7}{2}}\alpha^{p+\frac{7}{2}}, \tag{5.56}$$

where it is now clear that $p+\frac{1}{2}$ must be integral. This is in agreement with the general form of the similarity solutions of §3.

To determine the unknown constants $A_0, A_1, \ldots$ we must use the boundary condition (5.39) which gives

$$\tilde{v}_-(0+) \sim o(|\alpha|^{-n-1}) \quad \text{as} \quad \alpha \to 0 \tag{5.57}$$

and

$$\tilde{v}_-(0-) \sim o(|\alpha|^{-n-1}) \quad \text{as} \quad \alpha \to 0. \tag{5.58}$$

Expressing $\tilde{v}_-(0+)-\tilde{v}_-(0-)$ in terms of $E(\alpha)$ gives

$$\tilde{v}_-(0+)-\tilde{v}_-(0-)-\frac{Z^{(n)}}{\alpha^{n+1}} = \frac{A_0 + A_1\alpha + \ldots + A_{p+\frac{7}{2}}\alpha^{p+\frac{7}{2}}}{\alpha^6 S_-(\alpha)} \tag{5.59}$$

† Any singularity at $\alpha = 0$ is easily shown to be removable.

and since $S_-(0) = 1$, we have, as $\alpha \to 0$

$$-\frac{Z^{(n)}}{\alpha^{n+1}} \sim A_0\alpha^{-6} + A_1\alpha^{-5} + \ldots + A_{p+\frac{7}{2}}\alpha^{p-\frac{5}{2}}. \tag{5.60}$$

Consider first the $n = 0$ problem. Comparing the two sides of (5.60) gives

$$A_0 = A_1 = \ldots = A_4 = 0, \tag{5.61}$$

$$p = +\tfrac{3}{2}, \tfrac{5}{2}, \tfrac{7}{2}, \ldots, \tag{5.62}$$

and

$$A_5 = -Z^{(0)}. \tag{5.63}$$

But the Kutta condition implies that the $n = 0$ component of the $\frac{1}{3}$ layer solution is regular at $\eta = 0$. Thus we must have $Z^{(0)} = 0$, which implies that $d_T = d_B$. Then (5.18) shows that

$$d = d_T = d_B \tag{5.64}$$

and the unique solution is $w_0 = 0$, $v_0 = d$, as we conjectured.

Consider next the $n = 1$ problem. Then (5.60) gives

$$A_0 = \ldots = A_3 = 0, \tag{5.65}$$

$$p = \tfrac{1}{2}, \tfrac{3}{2}, \ldots \tag{5.66}$$

and

$$A_4 = -Z^{(1)}. \tag{5.67}$$

We have seen that the Kutta condition permits an $|\eta|^{-\frac{1}{2}}$ singularity in the $\eta = 1$ component of the $\frac{1}{3}$ layer velocity field, but excludes stronger singularities. Thus we must take $p = \frac{1}{2}$ and reject the larger values. The solution is now uniquely determined and, after some algebra, we find, for $0 < z < h_T$

$$2\pi\tilde{v} = -\frac{\nu^{\frac{1}{12}}}{\alpha^2}[V'(0) - V_T'(0)] + \frac{2\Omega\alpha\nu^{\frac{1}{12}}[V_T'(0) - V_B'(0)]}{(1 - e^{\alpha^3 h_T/\Omega})\, S_+(\alpha)}\,(e^{\alpha^3 z/2\Omega} + e^{\alpha^3 h_T/\Omega}\, e^{-\alpha^3 z/2\Omega}), \tag{5.68}$$

and for $-h_B < z < 0$

$$2\pi\tilde{v} = -\frac{\nu^{\frac{1}{12}}}{\alpha^2}[V'(0) - V_B'(0)] + \frac{2\Omega\alpha\nu^{\frac{1}{12}}[V_T'(0) - V_B'(0)]}{(1 - e^{-\alpha^3 h_B/\Omega})\, S_+(\alpha)}\,(e^{\alpha^3 z/2\Omega} + e^{-\alpha^3 h_B/\Omega}\, e^{-\alpha^3 z/2\Omega}). \tag{5.69}$$

We can find v_1 (and hence w_1) by inverting the Fourier transform. The resulting integral can be evaluated by residues, but we will not give the details.

It appears at first sight that, according to (5.68) and (5.69) $\tilde{v}$ behaves like α^{-2} as $\alpha \to 0$, which would contradict our assumptions. However, if we expand, say, (5.68) about $\alpha = 0$ we find that

$$\tilde{v} \sim -\frac{\nu^{\frac{1}{12}}}{\alpha^2 h}(V'(0)\,h - V_T'(0)\,h_T - V_B'(0)\,h_B), \tag{5.70}$$

so that the regularity of $\tilde{v}$ at $\alpha = 0$ is guaranteed by (5.22).

The problem of finding w_2 and v_2 is complicated by the delta function singularity of w_2 at $\eta = 0$, $z = 0$. The velocity satisfies equation (5.31) and the boundary conditions (5.32) and (5.33), and because v_0 is constant, the boundary condition (5.12) shows that $w_2 = 0$ on $z = 0$, $\eta < 0$; so that (5.34) still holds for the $n = 2$ field. However, $\partial v/\partial z$ is now not defined at $z = 0$, $\eta = 0$ and the condition $\partial v(\eta, 0+)/\partial z = \partial v(\eta, 0-)/\partial z$ for all η (used in deriving (5.44) for the $n = 0$ and $n = 1$ fields) must be replaced for the $n = 2$ field by

$$\frac{\partial v}{\partial z}(\eta, 0+) - \frac{\partial v}{\partial z}(\eta, 0-) = \frac{M\nu^{-\frac{1}{3}}}{2\Omega}\delta'''(\eta), \tag{5.71}$$

where M is given by (5.29). Moreover, we cannot use $\partial\tilde{v}_+(0)/\partial z$ as the 'plus function' in forming the Wiener–Hopf equation, but we use instead the transform of

$$\tfrac{1}{2}\{(\partial v/\partial z)\,(\eta, 0+) + (\partial v/\partial z)\,(\eta, 0-)\}$$

which is zero for $\eta \leqslant 0$ and is well defined for $\eta > 0$. We call this function $\tilde{F}_+$. Then using (5.42) and (5.43), we find that the equation analogous to (5.47) is

$$\tilde{v}_-(0+) - \tilde{v}_-(0-) - \frac{Z^{(2)}}{\alpha^3} = -\tilde{F}_+ \frac{2\Omega \sinh\,[\alpha^3(h_T + h_B)/2\Omega]}{\alpha^3 \sinh\,(\alpha^3 h_T/2\Omega)\sinh\,(\alpha^3 h_B/2\Omega)} + \frac{\mathrm{i}M\nu^{-\frac{1}{3}}}{4\pi}\frac{\sinh\,[\alpha^3(h_T - h_B)/2\Omega]}{\sinh\,(\alpha^3 h_T/2\Omega)\sinh\,(\alpha^3 h_B/2\Omega)}. \tag{5.72}$$

The Wiener–Hopf problem (5.72) can now be solved as for the $n = 0, 1$ fields, although the extra term complicates the calculation, and the $n = 2$ field can be obtained. We shall not give the details.

A special case of this problem which has been examined experimentally (Hide & Titman 1967) is when $U = 0$ and $Q = 1$, so that the disk rotates with angular velocity $\Omega(1+\epsilon)$ between rigid parallel plane walls. From (5.25), (5.26) and (5.27) we find

$$V_T'(0) - V_B'(0) = \frac{\Omega}{2}\frac{\epsilon a h p(p_T - p_B)}{h_T p_T + h_B p_B + hp} \tag{5.73}$$

so that, in view of (5.68) and (5.69) we have the result that the recirculating velocities of $O(\nu^{-\frac{5}{12}})$ still dominate the $\frac{1}{3}$ layer, unless $h_T = h_B$. In this latter case, the velocities of the $\frac{1}{3}$ layer are $O(\nu^{-\frac{1}{3}})$. Moreover, the problem of finding the structure of the $\frac{1}{3}$ layers is greatly simplified, since, by the symmetry, $\partial v/\partial z = 0$ on $z = 0$ for all $\eta(\neq 0)$ and we no longer have to solve a Wiener–Hopf problem. The solution is very similar to that given in §4 for the 'split disk' if we put $\epsilon = 0$; the change is due to the different p in the outer $\frac{1}{4}$ layer.

The most striking feature of the velocity field generated by the rising disk is the recirculating nature of the velocity field in the $\frac{1}{3}$ layer. As we have stressed, the velocity field in the $\frac{1}{3}$ layer is greater by a factor $O(\nu^{-\frac{1}{12}})$ than that required to conserve mass. It is the geometry of the edge of the disk which, by allowing weaker singularities in the $\frac{1}{3}$ layer dependence on η, enables the strength of the velocity field to be larger without violating the Kutta condition.

Physical discussion

The analysis of this section is complex and for the benefit of the reader who does not wish to follow the detailed mathematical analysis we will now summarize the important steps and discuss them in a more physical way. The disk of radius a is rising with velocity U between two planes h apart. In the Taylor column ahead of and behind the disk, there are swirl velocities v_G and axial velocities w_G of order

$$v_G \sim UE^{-\frac{1}{2}} \quad w_G \sim U. \tag{5.74}$$

The Ekman number will be defined on the body size

$$E = \nu/\Omega a^2. \tag{5.75}$$

The radial velocity in the Taylor column is zero. Outside the Taylor column, the geostrophic flow is zero. We allow for the possibility that the disk has a rotation speed $UE^{-\frac{1}{2}}/a$, in order that the torque vanish.

The Taylor column is bounded at the horizontal ends by Ekman layers of thickness $(\nu/\Omega)^{\frac{1}{2}} = \delta_{\frac{1}{2}}$, say, in which the velocity falls to the value on the walls. It is bounded on the sides by Stewartson layers which have a complex sandwich structure. There are two inner layers and one outer layer, each of thickness

$$\delta_{\frac{1}{4}} = a(h/a)^{\frac{1}{2}}\, E^{\frac{1}{4}}. \tag{5.76}$$

We shall retain the dependence on a and h separately, to cover the situations in which a/h is large or small. The outer layer goes from the top to the bottom outside $r = a$, and the two inner layers go from the body to the top and bottom inside the circumscribing cylinder. There is a single central layer, of thickness

$$\delta_{\frac{1}{3}} = a(h/a)^{\frac{1}{3}}\, E^{\frac{1}{3}} \tag{5.77}$$

extending from the top to the bottom.

In the $\frac{1}{4}$ layers, only the viscous term $\nu\, \partial^2 v/\partial r^2$ becomes important and the flow is 'quasi-geostrophic', with the swirl velocity v and radial velocity u independent of height z, and the axial velocity still small compared with the swirl velocity. In these layers, the orders of magnitude are

$$v_{\frac{1}{4}} \sim UE^{-\frac{1}{2}}, \quad w_{\frac{1}{4}} \sim (a/h)^{\frac{1}{2}}\, E^{\frac{1}{4}} v_{\frac{1}{4}} \sim (a/h)^{\frac{1}{2}}\, UE^{-\frac{1}{4}}, \quad u_{\frac{1}{4}} \sim (a/h)\, E^{\frac{1}{2}} v_{\frac{1}{4}} \sim (a/h)\, U. \tag{5.78}$$

These flows are still driven by the Ekman suction and are completely determined when the swirl velocities are specified on the central edges of the layers, i.e. on the cylinder $r = a$. The $\frac{1}{4}$ layer velocities cannot be analytic across $r = a$ and the discontinuities in velocities and velocity derivatives are smoothed out in the $\frac{1}{3}$ layer, in which the viscous term $\nu\, \partial^2 w/\partial r^2$ is also important and the fluctuations in v and w are comparable.

It is clear from (5.78) that fluid flows into the $\frac{1}{3}$ layer at the rate $U^2 a$. In order to distribute this flux through the central layer and to carry it around the body, it is clear that axial velocities of order

$$w^{(2)}_{\frac{1}{3}} \sim Ua/\delta_{\frac{1}{3}} = U(a/h)^{\frac{1}{3}}\, E^{-\frac{1}{3}} \tag{5.79}$$

are necessary, with which are associated swirl and radial velocities of order

$$v^{(2)}_{\frac{1}{3}} \sim Ua/\delta_{\frac{1}{3}}, \quad u^{(2)}_{\frac{1}{3}} \sim Ua/h. \tag{5.80}$$

We can now demonstrate heuristically that the velocities (5.79) and (5.80) cannot accommodate the jump in the $\frac{1}{4}$ layer velocities, so that arguments based on flux conservation are insufficient to determine the qualititative order of the central layer. In general, the $\frac{1}{4}$ layer swirl velocity will have jumps in v, $\mathrm{d}v/\mathrm{d}r$ and $\mathrm{d}^2v/\mathrm{d}r^2$ of order

$$[v_{\frac{1}{4}}] \sim UE^{-\frac{1}{2}}, \quad \left[\frac{\mathrm{d}}{\mathrm{d}r} v_{\frac{1}{4}}\right] = \frac{UE^{-\frac{1}{2}}}{\delta_{\frac{1}{4}}} = \frac{U}{(ah)^{\frac{1}{2}}}\, E^{-\frac{3}{4}}, \quad \left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} v_{\frac{1}{4}}\right] = \frac{UE^{-\frac{1}{2}}}{\delta^2_{\frac{1}{4}}} = \frac{UE^{-1}}{ah}. \tag{5.81}$$

The possible changes in these quantities due to the $\frac{1}{3}$ velocity given by (5.80) are

$$[v_{\frac{1}{3}}] = \frac{Ua}{\delta_{\frac{1}{3}}} = U\left(\frac{a}{h}\right)^{\frac{1}{3}} E^{-\frac{1}{3}}, \quad \left[\frac{\mathrm{d}}{\mathrm{d}r} v_{\frac{1}{3}}\right] = \frac{Ua}{\delta^2_{\frac{1}{3}}} = \frac{UE^{-\frac{2}{3}}}{a^{\frac{1}{3}} h^{\frac{2}{3}}}, \quad \left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} v_{\frac{1}{3}}\right] = \frac{Ua}{\delta^3_{\frac{1}{3}}} = \frac{UE^{-1}}{ah}. \tag{5.82}$$

The jumps in the second derivatives match exactly or equivalently $u_{\frac{1}{3}} \sim u_{\frac{1}{4}}$. This is not an accident, because the vertical flux in the $\frac{1}{3}$ layer is proportional to the jump across it of the second derivative of the quasi-geostrophic flow, and the orders of magnitude were picked so that the flux would balance. The jumps in the velocity and velocity gradient across the $\frac{1}{3}$ layer are too small, and hence for consistency we would require that $v_{\frac{1}{4}}$ and $\mathrm{d}v_{\frac{1}{4}}/\mathrm{d}r$ be

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

continuous across the $\frac{1}{3}$ layer. However, in the present problem this requirement cannot be satisfied (although it always can in the problems that have so far been reported on in the literature), because it provides four equations in only three unknowns, namely the three unknown swirl velocities at the central edges of the $\frac{1}{4}$ layers.

A jump in $\mathrm{d}v_{\frac{1}{4}}/\mathrm{d}r$, which physically implies a local tangential stress on the $\frac{1}{3}$ layer, can be accommodated by a velocity field of magnitude

$$v_{\frac{1}{3}}^{(1)} \sim w_{\frac{1}{3}}^{(1)} \sim U(a/h)^{\frac{1}{6}} E^{-\frac{5}{12}}, \quad u_{\frac{1}{3}}^{(1)} \sim U(a/h)^{\frac{5}{6}} E^{-\frac{1}{6}}. \tag{5.83}$$

The jump in $v_{\frac{1}{3}}^{(1)}$ is less than $[v_{\frac{1}{4}}]$, so that we still require $v_{\frac{1}{4}}$ to be continuous across the $\frac{1}{3}$ layer. A further condition is obtained from the requirement that the total tangential force (more precisely couple) on the $\frac{1}{3}$ layer vanish; this gives (5.22) and we now have the correct number of three equations to determine the $\frac{1}{4}$ layers. The objection to (5.83), and this is why order of magnitude arguments are sometimes dangerous and precise analysis is needed, is that it gives a jump in the second derivatives larger than that allowed by (5.81). Equivalently, $u_{\frac{1}{3}}^{(1)}$ is larger than $u_{\frac{1}{4}}$ or the vertical mass flux is of order $E^{-\frac{1}{6}}$. However, this difficulty is apparent rather than real because the solution can be chosen (as follows from the detailed analysis) so that $u_{\frac{1}{3}}^{(1)}$ vanishes at the edge of the $\frac{1}{3}$ layer, and it also follows that the vertical mass flux from the velocity $w_{\frac{1}{3}}^{(1)}$ is zero.

These physical arguments which establish the qualitative structure of the shears are however incomplete for reasons which appear when the detailed structure of the $\frac{1}{3}$ layer is investigated by mathematical analysis. It appears then that $\frac{1}{3}$ layer solutions can be constructed to match any external $\frac{1}{4}$ layers; in other words, the dynamics of the $\frac{1}{3}$ layer do not impose any restriction on the $\frac{1}{4}$ layer velocities at the central edges. In particular, a $\frac{1}{3}$ layer solution can be constructed to smooth out discontinuities in $v_{\frac{1}{4}}$ across the central layer. The physical argument just given fixes the orders of magnitude essentially by saying that the velocities should be as small as possible. Thus a discontinuity in $v_{\frac{1}{4}}$ would require axial $\frac{1}{3}$ layer velocities $w_{\frac{1}{3}}^{(0)} \sim UE^{-\frac{1}{2}}$. However, such a plausible assumption still leaves the further difficulty that the $\frac{1}{3}$ layer solutions are not unique and an infinite number exist to match any $\frac{1}{4}$ layer flow field. To make the solution unique it is necessary to say something about the singularity of the $\frac{1}{3}$ layer equations at the edge $r = a$, $z = 0$.

The trouble has of course arisen because the assumptions that give the $\frac{1}{3}$ layer are not valid near the edge and the structure of the Ekman layer controls the permitted singularity and thus the solution needs to be completed by matching the $\frac{1}{3}$ layer and the Ekman layer at the edge. However, this matching is difficult and so far has defined attack, and was therefore bypassed by a plausible physical hypothesis which we called a Kutta condition, and which in effect states that the singularity should be as weak as possible. It can be shown from a solution of the Wiener–Hopf problem for the $\frac{1}{3}$ layer that a discontinuity in $v_{\frac{1}{4}}$ would lead to a singularity like $(r-a)^{-\frac{3}{2}}$ in $v_{\frac{1}{3}}$, so we drop it and just retain the $(r-a)^{-\frac{1}{2}}$ singularity which occurs when $v_{\frac{1}{4}}$ is continuous but $\mathrm{d}v_{\frac{1}{4}}/\mathrm{d}r$ is not, and the $(r-a)^{-1}$ singularity associated with the jump in $\mathrm{d}^2v_{\frac{1}{4}}/\mathrm{d}r^2$. In problems of this kind, it is sometimes possible to fix the singularity by saying that the solution should be square integrable, but this does not work in the present problem.

It is clear that the analysis applies with minor modification to an axisymmetric lenticular body.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

6. The rising sphere

We replace the disk $0 \leqslant r \leqslant a$, $z = 0$ of §5 by a sphere

$$z = \pm f(r) = \pm (a^2 - r^2)^{\frac{1}{2}}, \tag{6.1}$$

where $a < \min(h_T, h_B)$.

The geostrophic motion vanishes for $r > a$, and for $r < a$ we find that, for $z > 0$,

$$V_{GT}(r) = -r\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}}\left\{U\left(1-\frac{r^2}{a^2}\right)^{\frac{1}{4}} - \epsilon\Omega\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\right\}\Big/\left\{Q\left(1-\frac{r^2}{a^2}\right)^{\frac{1}{4}}+1\right\}, \tag{6.2}$$

and for $z < 0$

$$V_{GB}(r) = r\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}}\left\{U\left(1-\frac{r^2}{a^2}\right)^{\frac{1}{4}} + \epsilon\Omega\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\right\}\Big/\left\{\left(1-\frac{r^2}{a^2}\right)^{\frac{1}{4}}+1\right\}. \tag{6.3}$$

The more complicated r-dependence of the geostrophic interior is due to the variation with position of the thickness of the Ekman layer on the sphere. Moreover, U and $\epsilon\Omega$ contribute different forms of swirl to the geostrophic interior and, in particular, when $\epsilon = 0$ and $Q = 1$ (as for a sphere rising freely between plane rigid end walls), $V_{GB}(a) = V_{GT}(a) = 0$. We can anticipate that the non-rotating sphere will produce a weaker shear layer than the rotating one and this makes a general discussion awkward. Thus we deal with the problem in two stages.

(i) *Sphere rising without rotation between rigid end walls*

The outer $\frac{1}{4}$ layer is identical to that for a disk and so

$$V = B\,\mathrm{e}^{-p\xi} \quad (\xi > 0, \quad -h_B \leqslant z \leqslant h_T). \tag{6.4}$$

For $r < a$ we have, where $h(r) = h_T - (a^2 - r^2)^{\frac{1}{2}}$,

$$-\frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} V - \frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}} \frac{V}{(1-r^2/a^2)^{\frac{1}{4}}} + \frac{\nu h(r)}{2\Omega}\frac{\mathrm{d}^2 V}{\mathrm{d}r^2} = \frac{Ua}{2}. \tag{6.5}$$

The first term on the left represents the Ekman flux on the rigid upper plate and it is clearly negligible compared to the second term on the left, which is the Ekman flux on the sphere. Thus the structure of the inner shear layer is determined to leading order solely by the Ekman layer on the sphere and it is independent of the nature of the upper boundary. We see that the terms in (6.5) balance if $\partial/\partial r \sim \nu^{-\frac{2}{7}}$ and if we write

$$\zeta = \frac{a-r}{\nu^{\frac{2}{7}}} = -\frac{\xi}{\nu^{\frac{1}{28}}} = -\eta\nu^{\frac{1}{21}}, \tag{6.6}$$

and retain only the largest terms in (6.5), we have

$$\frac{h_T}{2\Omega}\frac{\mathrm{d}^2 V_T}{\mathrm{d}\zeta^2} - \left(\frac{a}{2\zeta}\right)^{\frac{1}{4}}\frac{1}{2\Omega^{\frac{1}{2}}}V_T = \tfrac{1}{2}Ua\nu^{-\frac{3}{7}} + O(\nu^{-\frac{5}{14}}), \tag{6.7}$$

where the error is due to the neglect of the Ekman flux on $z = h_T$. The errors inherent in (6.5) and the error introduced by replacing $h(r)$ by h_T are of smaller order of magnitude.

To solve (6.7) we follow Stewartson (1967) and write

$$\zeta = q_T^{-1}s \quad \text{and} \quad V_T = \nu^{-\frac{3}{7}}\mu_T F(s) \tag{6.8}$$

where

$$q_T^{-1} = h_T^{\frac{4}{7}}2^{\frac{1}{7}}/\Omega^{\frac{2}{7}}a^{\frac{1}{7}} \quad \text{and} \quad \mu_T = -2^{\frac{2}{7}}Ua^{\frac{5}{7}}h_T^{\frac{1}{7}}\Omega^{\frac{3}{7}}. \tag{6.9}$$

Then (6.7) becomes
$$F''(s)-F(s)/s^{\frac{1}{4}}=-1. \tag{6.10}$$
Let us define a solution $F_0(s)$ of the homogeneous equation
$$F_0''(s)-F_0(s)/s^{\frac{1}{4}}=0 \quad \text{with} \quad F_0(0)=1,\ F_0(\infty)=0. \tag{6.11}$$
Then F_0 is like the decaying exponential $e^{p\xi}$ in the interior shear layer of the disk. We define a solution $G(s)$ of the inhomogeneous equation (6.10) by the requirements that
$$G(s)\sim s^{\frac{1}{4}} \quad \text{as} \quad s\to\infty \qquad G(0)=0. \tag{6.12}$$
This makes $\nu^{-\frac{3}{7}}\mu_T G(q_T\zeta)\sim V_{GT}(r)$ as $\zeta\to\infty$, so that G is like the constant part of the interior shear layer solution for the disk.

We can find F_0 explicitly by transforming (6.10) into Bessel's equation and we get
$$F_0(s)=cs^{\frac{1}{2}}K_{\frac{4}{7}}(\tfrac{8}{7}s^{\frac{7}{8}}), \tag{6.13}$$
where c is a numerical constant. Thus for large s, F_0 behaves like
$$s^{\frac{1}{16}}\exp(-\tfrac{8}{7}s^{\frac{7}{8}}), \tag{6.14}$$
so that F_0 decays rapidly with increasing s.

We can find G in closed form by variation of parameters, but we will not write down the rather cumbersome results, since we can get enough information to see how the matching goes without using the explicit solutions.

Using the well-defined functions F_0 and G, we can write the general solution for the $\frac{2}{7}$ layer in a by now familiar form,
$$V_T=\nu^{-\frac{3}{7}}\mu_T\{G(q_T\zeta)+A_TF_0(q_T\zeta)\}, \tag{6.15}$$
where A_T is an unknown constant, whose dependence on ν needs to be determined as it is not obviously $O(1)$. The region in which (6.15) holds is called a $\frac{2}{7}$ layer, because its width is $O(\nu^{\frac{2}{7}})$.

The next step is to expand both the $\frac{1}{4}$ and the $\frac{2}{7}$ layer for small values of their respective arguments; then expressing the results in terms of η and invoking the matching principle (4.15) will show us what sort of $\frac{1}{3}$ layer is needed to provide a match, and determine B, A_T and the corresponding unknown A_B for the $\frac{2}{7}$ layer below the sphere.

For small s the differential equation shows that
$$F_0(s)=1+ks+\tfrac{16}{21}s^{\frac{7}{4}}+\frac{16k}{77}s^{\frac{11}{4}}+\dots \tag{6.16}$$
and
$$G(s)=k's-\tfrac{1}{2}s^2+\dots. \tag{6.17}$$
To find the numerical values of k and k' we need the explicit solutions.

Thus, expanding (6.4) and (6.15) we find that
$$V=B(1-\nu^{\frac{1}{12}}p\eta+\tfrac{1}{2}\nu^{\frac{1}{6}}p^2\eta^2+\dots) \tag{6.18}$$
$$V_T=\nu^{-\frac{3}{7}}\mu_T\{A_T-\nu^{\frac{1}{21}}q_T\eta(kA_T+k')+\nu^{\frac{1}{12}}q_T^{\frac{7}{4}}(-\eta)^{\frac{7}{4}}(\tfrac{16}{21}A_T)+\tfrac{1}{2}\nu^{\frac{2}{21}}q_T^2\eta^2+\dots\} \tag{6.19}$$
and similarly for $z<0$, replacing T by B but taking μ_B positive,
$$V_B=\nu^{-\frac{3}{7}}\mu_B\{A_B-\nu^{\frac{1}{21}}q_B\eta(kA_B+k')+\nu^{\frac{1}{12}}q_B^{\frac{7}{4}}(-\eta)^{\frac{7}{4}}(\tfrac{16}{21}A_B)+\tfrac{1}{2}\nu^{\frac{2}{21}}q_B^2\eta^2+\dots\}. \tag{6.20}†$$

† The relative error in (6.7) and hence in (6.19) and (6.20) from which it is derived is $\nu^{\frac{1}{14}}$, so that it might appear that only the first two terms in (6.19) and (6.20) are significant. However, consideration of the more accurate form of (6.7), which retains the first term on the left in (6.5), shows that the solution for small s is of the same form as (6.19) and (6.20), that is no new powers of s arise. Thus the coefficients of the various powers of η in the expansions (6.19) and (6.20) are each correct with relative error $\nu^{\frac{1}{14}}$.

The equations (6.18), (6.19) and (6.20) provide boundary conditions for the central $\frac{1}{3}$ layer as $\eta \to \pm\infty$. Again for this region we split v and w into components

$$v = \Sigma v^{(n)}, \quad w = \Sigma w^{(n)}, \tag{6.21}$$

where $v^{(n)}$, $w^{(n)}$ satisfy the boundary conditions (5.35), (5.36) and (5.37) and the coefficients $d^{(n)}$ follow from (6.18), (6.19) and (6.20). Note that fractional powers of n appear, in particular $n = \frac{7}{4}$. It may seem physically artificial to select the $\frac{1}{3}$ layer components according to their dependence on η at infinity, but mathematically it is natural since it is the dependence on large η that determines the solution. Also, the ν dependence is not known yet because A_T, A_B and B have to be found. However, we can anticipate that the η ordering will more or less correspond so that $v^{(0)}$ will be the largest component, and so on.

The Ekman condition on $z = -h_B$, h_T is

$$w = \pm \frac{\nu^{\frac{1}{6}}}{2\Omega^{\frac{1}{2}}} \frac{\partial}{\partial \eta} v \tag{6.22}$$

with completely negligible error $O(\nu^{\frac{1}{2}})$. On the sphere, the geometry complicates the relation and we have

$$\left.\begin{aligned} w(\eta, z) &= \pm \frac{\nu^{\frac{1}{12}}}{2\Omega^{\frac{1}{2}}} \frac{\partial}{\partial \eta} \{(-\eta)^{-\frac{1}{4}} v(\eta, z)\} + U \\ \text{on} \quad z &= \pm 2a^{\frac{1}{2}} \nu^{\frac{1}{6}} (-\eta)^{\frac{1}{2}} \quad (-\infty < \eta < 0), \end{aligned}\right\} \tag{6.23}$$

where we have made the boundary-layer approximation to the geometry of the sphere. Thus

$$w(\eta, \pm 0) = \pm \frac{\nu^{\frac{1}{12}}}{2\Omega^{\frac{1}{2}}} \frac{\partial}{\partial \eta} \{(-\eta)^{\frac{1}{4}} v(\eta, \pm 0)\}, \tag{6.24}$$

with a relative error $O(\nu^{\frac{1}{6}})$.

Consider now the problem for $v^{(0)}, w^{(0)}$. This is identical with the same problem for the disk. The reason is that the coordinate stretching for the $\frac{1}{3}$ layer flattens the sphere so that (at least to leading order) the $\frac{1}{3}$ layer 'sees' a disk and a sphere in the same way. The analysis for the disk applies and we conclude that if $v^{(0)} \neq$ constant and $w^{(0)} \neq 0$, then $v^{(0)} \sim \eta^{-\frac{3}{2}}$ at the edge. We now apply the Kutta condition to rule this possibility out. As Stewartson (1966) has shown, the $\frac{1}{3}$ layer joins on to the Ekman layer through a joining region whose lateral scale is of order $\nu^{\frac{2}{5}}$. At the joining region,

$$v^{(0)} \sim \nu^{-\frac{3}{7}} A_T (\nu^{\frac{2}{5}}/\nu^{\frac{1}{3}})^{-\frac{3}{2}} = A_T \nu^{-\frac{37}{70}}$$

and this is unacceptable unless A_T is $O(\nu^{\frac{1}{35}})$. The last condition can be shown to lead to inconsistencies and we conclude that the $v^{(0)}, w^{(0)}$ field cannot have a singularity and the swirl velocity is continuous (to leading order) across the third layer, i.e.

$$w^{(0)} = 0, \quad v^{(0)} = B = \mu_T \nu^{-\frac{3}{7}} A_T = \mu_B \nu^{-\frac{3}{7}} A_B. \tag{6.25}$$

A further relation follows from considering the $v^{(1)}$, $w^{(1)}$ component. Again the equations and boundary conditions are identical with the same order component of the disk problem. The Kutta condition (or with perhaps equal plausibility the hypothesis of minimum singularity) fixes the singularity at the edge. Again the total stress across the $\frac{1}{3}$ layer must be continuous and we have

$$h_T \mu_T q_T (kA_T + k') + h_B \mu_B q_B (kA_B + k') = Bp\nu^{\frac{39}{84}}. \tag{6.26}$$

From (6.25) and (6.26) we deduce A_T, A_B and B, and find that A_T and A_B are $O(\nu^0)$. Thus the inner $\frac{2}{7}$ layer and outer $\frac{1}{4}$ layer are now determined to leading order. Note that if $h_T = h_B$, $A_T = A_B = B = 0$. Thus in the symmetric position the external $\frac{1}{4}$ layer vanishes. However, the inner $\frac{2}{7}$ layer is still there, being provided entirely by $G(q\zeta)$, and the orders of magnitude and general structure of the $\frac{1}{3}$ layer are unaffected.

The detailed form of $v^{(1)}$ and $w^{(1)}$ is given by the analysis of § 5 on inserting the appropriate values of $d^{(1)}$, $d_T^{(1)}$, $d_B^{(1)}$. Again, this field carries no net flux and to leading order the $\frac{1}{3}$ layer is a closed recirculating system, driven by the discontinuity of local tangential stress, and with an $\eta^{-\frac{1}{2}}$ singularity at the equator. Note that since $\nu^{\frac{1}{12}} \ll \nu^{\frac{1}{21}}$, the Ekman condition (6.24) is such that the right-hand side does not affect the equation for $v^{(1)}$ and $w^{(1)}$ and the geometry of the sphere has not yet entered the shear-layer structure.

We can now obtain equations for $v^{(\frac{7}{4})}, w^{(\frac{7}{4})}$ (which also recirculates mass), and the field $v^{(2)}$, $w^{(2)}$ which carries the mass flux and which from the matching imposed by (6.19) and (6.20) is seen to be $O(\nu^{-\frac{1}{3}})$. Indeed the $n = 2$ field can be shown to satisfy the same equations as for the rising disk and its structure is closely similar. However, the Wiener–Hopf solution obtained in § 5 will not provide the solution to the $n = \frac{7}{4}$ field, which from (6.19) and (6.20) is $O(\nu^{-\frac{29}{84}})$, because the Ekman condition (6.24) gives

$$w^{(\frac{7}{4})}(\eta, \pm 0) = \pm \frac{\nu^{\frac{1}{12}}}{2\Omega^{\frac{1}{2}}} \frac{\partial}{\partial \eta} \{(-\eta)^{-\frac{1}{4}} v^{(0)}\} \quad \text{for} \quad \eta < 0, \tag{6.27}$$

where the right-hand side is $O(\nu^{-\frac{29}{84}})$ on substituting (6.25). The problem to determine the $n = \frac{7}{4}$ field is indeed somewhat more difficult than for $n = 0, 1, 2$ and we have not obtained the solution. However, the order of magnitude is given by the matching argument. It follows from (6.27) that in the vicinity of the equator, the $n = \frac{7}{4}$ velocities have singularities like $\eta^{-\frac{5}{4}}$.

To summarize the results so far, we have an inner layer of width

$$\delta_{\frac{2}{7}} = a(h/a)^{\frac{4}{7}} E^{\frac{2}{7}}, \tag{6.28}$$

in which the swirl and radial velocities are independent of z, and the axial velocity is linear in z, and the orders of magnitude are

$$v_{\frac{2}{7}} = U(h/a)^{\frac{1}{7}} E^{-\frac{3}{7}}, \quad w_{\frac{2}{7}} = (a/h)^{\frac{5}{7}} E^{\frac{1}{7}} v_{\frac{2}{7}}, \quad u_{\frac{2}{7}} = (a/h)^{\frac{8}{7}} E^{\frac{3}{7}} v_{\frac{2}{7}}. \tag{6.29}$$

The swirl velocity is smaller than that in the geostrophic interior because the geostrophic velocity vanishes at $r = a$. There is an outer layer of width

$$\delta_{\frac{1}{4}} = a(h/a)^{\frac{1}{2}} E^{\frac{1}{4}}, \tag{6.30}$$

in which the velocities are

$$v_{\frac{1}{4}} = U(h/a)^{\frac{1}{7}} E^{-\frac{3}{7}}, \quad w_{\frac{1}{4}} = (a/h)^{\frac{1}{2}} E^{\frac{1}{4}} v_{\frac{1}{4}}, \quad u_{\frac{1}{4}} = (a/h)\, E^{\frac{1}{2}} v_{\frac{1}{4}}. \tag{6.31}$$

The leading order motion (apart from the constant swirl) in the central $\frac{1}{3}$ layer, which has width

$$\delta_{\frac{1}{3}} = a(h/a)^{\frac{1}{3}} E^{\frac{1}{3}}, \tag{6.32}$$

is a recirculating eddy in which the velocities are

$$v_{\frac{1}{3}} \sim w_{\frac{1}{3}} \sim U(a/h)^{\frac{2}{21}} E^{-\frac{8}{21}}, \quad u_{\frac{1}{3}} \sim (a/h)^{\frac{2}{3}} E^{\frac{1}{3}} v_{\frac{1}{3}}. \tag{6.33}$$

The mass flux in the $\frac{1}{3}$ layer is carried by the smaller velocity of order $Ua/\delta_{\frac{1}{3}}$. Also, the leading order flow has an inverse square root singularity where the $\frac{1}{3}$ layer meets the Ekman layer.

Note that the structure of the shear layers is independent to leading order of the nature of the upper surface, because the leading order boundary condition on the upper surface is $w = 0$.

(ii) *Sphere rotating without rising*

In the special case $h_T = h_B$ and $Q = 1$ (upper surface rigid) this problem was studied by Stewartson (1967). We shall consider the general case since, just as for the disk, the symmetric situation is atypical. In fact, the symmetric case is similar to the split disk problem considered in §4.

The outer layer is still given by (6.4) replacing B by D. The inner layer is now given by

$$\frac{h_T}{2\Omega}\frac{\mathrm{d}^2V_T}{\mathrm{d}\zeta^2}-\left(\frac{a}{2\zeta}\right)^{\frac{1}{4}}\frac{1}{2\Omega^{\frac{1}{2}}}(V_T-\epsilon\Omega a) = O(V_T\nu^{\frac{1}{14}}), \tag{6.34}$$

so that

$$V_T = \epsilon\Omega a + C_T F_0(s) \tag{6.35}$$

where $F_0(s)$ is given by (6.13). There is a similar equation for V_B.

Expanding to find matching conditions for the $\frac{1}{3}$ layer, we find

$$V = D(1-\nu^{\frac{1}{12}}p\eta+\tfrac{1}{2}\nu^{\frac{1}{6}}p^2\eta^2+\ldots), \tag{6.36}$$

$$V_T = \epsilon\Omega a + C_T(1-\nu^{\frac{1}{21}}q_Tk\eta+\tfrac{16}{21}\nu^{\frac{1}{12}}q_T^{\frac{7}{4}}(-\eta)^{\frac{7}{4}}+O(\eta^{\frac{11}{4}})), \tag{6.37}$$

and

$$V_B = \epsilon\Omega a + C_B(1-\nu^{\frac{1}{21}}q_Bk\eta+\tfrac{16}{21}\nu^{\frac{1}{12}}q_B^{\frac{7}{4}}(-\eta)^{\frac{7}{4}}+O(\eta^{\frac{11}{4}})). \tag{6.38}$$

Thus the $\frac{1}{3}$ layer has velocity fields corresponding to $n = 0, 1, \frac{7}{4}, 2$, etc. The $n = 0, 1$ components clearly satisfy the same equations as for the non-rotating case, and we deduce, as before, continuity of velocity and total tangential stress. Then from (6.36), (6.37) and (6.38),

$$C_T+\epsilon\Omega a = C_B+\epsilon\Omega a = D \tag{6.39}$$

and

$$h_TC_Tq_Tk+h_BC_Bq_Bk = \nu^{\frac{1}{28}}Dhp. \tag{6.40}$$

To leading order,

$$D = \epsilon\Omega a, \quad C_T = C_B = \frac{\nu^{\frac{1}{28}}\epsilon\Omega ahp}{(h_Tq_T+h_Bq_B)k}. \tag{6.41}$$

Thus the outer layer now has a swirl velocity $O(\nu^{-\frac{1}{28}})$ larger than those in the inner $\frac{2}{7}$ layer.

The $n = 1$ field inside the $\frac{1}{3}$ layer is again given by the solution for the rising disk, with appropriate choices of the constants $d^{(1)}$. The essential feature is that the velocity field recirculates and is $O(\epsilon\nu^{\frac{1}{12}})$. Also the velocities have singularities like $\eta^{-\frac{1}{2}}$ at the equator. (In the special symmetrical case, $v^{(1)}$ is proportional to η and $w^{(1)} = 0$.) The Ekman condition (6.24) should now be

$$w(\eta, \pm 0) = \pm\frac{\nu^{\frac{1}{12}}}{2\Omega^{\frac{1}{2}}}\frac{\partial}{\partial\eta}\{(-\eta)^{\frac{1}{4}}[v(\eta, \pm 0)-\epsilon\Omega a]\}, \tag{6.42}$$

and clearly the right-hand side is unimportant for the $n = 0, 1$ fields.

The calculation of the $n = \frac{7}{4}$ field proceeds as for the non-rotating case. The velocities are now $O(\epsilon\nu^{\frac{5}{42}})$, and equation (6.42) has to be employed with $v = \epsilon\Omega a+C_T$ or $\epsilon\Omega a+C_B$ on the right-hand side. For the symmetric case, $w = 0$ generally for $r > a$, $z = 0$ and the $n = \frac{7}{4}$ field can be found without solving a Wiener–Hopf problem; Stewartson (1966) has given closed form expressions for the velocities.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The $n = 2$ field, which carries the mass flux, is not now given by the analysis previously carried out. There are two reasons. First, the $v^{(1)}$ field makes a contribution through the Ekman condition (6.42) of the same order in ν as that imposed by the matching. And secondly, the error in (6.36) also produces terms of the same order. The structure of the $n = 2$ field is not independent of the upper boundary condition because the higher approximation to (6.34) brings in Q. The velocities are $O(\epsilon\nu^{\frac{1}{6}})$.

To sum up for the case of rotation without translation, the variation of inner layer velocity is (writing explicitly the dependence on h)

$$v_{\frac{2}{7}} \sim \epsilon\Omega a(h/a)^{\frac{1}{14}} E^{\frac{1}{28}} \tag{6.43}$$

with the other velocities given by (6.29). The outer layer swirl velocity is

$$v_{\frac{1}{4}} \sim \epsilon\Omega a \tag{6.44}$$

with other components given by (6.31).

Inside the central layer, in addition to the uniform swirl, there are fluctuations

$$v_{\frac{1}{3}} \sim w_{\frac{1}{3}} \sim \epsilon\Omega a(a/h)^{\frac{1}{6}} E^{\frac{1}{12}}, \tag{6.45}$$

unless the body is midway between the end walls, in which case the relevant order is $\epsilon\Omega a E^{\frac{5}{42}}(a/h)^{\frac{29}{84}}$. Mass flux in the $\frac{1}{3}$ layer is carried by axial velocities of order $\epsilon\Omega a E^{\frac{1}{6}}(a/h)^{\frac{1}{3}}$. The mass flux carried by the layer is of order $\epsilon\Omega a^2 E^{\frac{1}{2}}$ per unit length of circumference.

Comparing these results with (6.29), (6.31) and (6.33) for the case of rising without rotation, and putting $\epsilon \sim E^{-\frac{1}{2}}$, we see that rotation generally produces velocities that are larger by $E^{-\frac{1}{28}}$ in the inner and central layer and $E^{-\frac{1}{14}}$ in the outer layer. In any real situation, these differences are of course academic.

7. A viscous Taylor column

The analysis of shear layers breaks down when the thickness of the layer is not small compared with the radius of the body. For the rising disk, the analysis required that $\delta_{\frac{1}{4}} \ll a$, i.e.

$$E^{\frac{1}{4}} \ll (a/h)^{\frac{1}{2}} \quad \text{or} \quad h/a \ll E^{-\frac{1}{2}}. \tag{7.1}$$

For the sphere, it is more appropriate to apply the restriction to the inner $\frac{2}{7}$ layer, giving

$$E^{\frac{2}{7}} \ll (a/h)^{\frac{4}{7}} \tag{7.2}$$

which, however, is essentially the same condition. It is also necessary if $h < a$ that h should be large compared with the Ekman layer thickness, i.e.

$$h/a \gg E^{\frac{1}{2}}. \tag{7.3}$$

The conditions (7.1) or (7.2) and (7.3) are necessary for the analysis of previous sections to apply. (The matter of sufficiency is more obscure, since not only must inertial effects be negligible but the stability of the flows is an open question.)

It is now interesting to see what happens as h/a is increased, keeping the Ekman number constant. When $h/a \sim E^{-\frac{1}{2}}$, the inner and outer layers are comparable in width to the body and the concept of shear layers is invalid. However, examination of the shear layer equations shows that the assumption of a thin inner and outer layer was used only in approximating derivatives $\partial/\partial r$ by $\partial/\partial x$ and replacing r by a, i.e. in employing a 'boundary layer' approximation to the geometry. The key step which led to the inner and outer layer was the introduction of the azimuthal viscous stress, while still neglecting both the axial viscous stress (which becomes important in the $\frac{1}{3}$ layer) and the radial stress (which is never important). Hence provided the $\frac{1}{3}$ layer remains thin, i.e.

$$\frac{\delta_{\frac{1}{3}}}{a} = \left(\frac{h}{a}\right)^{\frac{1}{3}} E^{\frac{1}{3}} \ll 1, \tag{7.4}$$

which is satisfied when $h/a \sim E^{-\frac{1}{2}}$ if $E^{\frac{1}{6}} \ll 1$, the velocity outside the $\frac{1}{3}$ layer is given by (for axisymmetrical flow)

$$v = V(r), \quad u = \frac{\nu}{2\Omega}\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{1}{r^2}\right)V, \quad w = -\frac{z}{r}\frac{d}{dr}(ru)+W(r). \tag{7.5}$$

It is readily verified that (7.5) gives an approximate solution of the equations of motion provided (7.4) is satisfied. The swirl velocity $V(r)$ is determined by mass continuity. Thus for a body of shape $z = f(r)$ rising with velocity U between plane walls, we have for $z > 0$, $r < a$

$$\frac{\nu(h_T - f(r))}{2\Omega}\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\right]V_T - \tfrac{1}{2}Q\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}V_T - \frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\frac{V_T}{[1+\{f'(r)\}^2]^{\frac{1}{4}}} = \tfrac{1}{2}Ur, \tag{7.6}$$

with similar equations for V_B in ($z < 0$, $r < a$) and V_E in ($r > a$).

It is clear that when $h/a \ll E^{-\frac{1}{2}}$, we can split up this equation into a geostrophic region and thin shear layers on $r = a$. For $h/a \sim E^{-\frac{1}{2}}$, the viscous term is never negligible, and we no longer have a geostrophic region in the flow. The azimuthal and radial (but not the axial) velocities are independent of height, and in this sense we can continue to speak about a Taylor column, but its interior is directly affected by viscous forces.

The solutions of (7.6) and the two similar equations cannot be analytic across $r = a$, and a $\frac{1}{3}$ layer still suffices to match up the velocities and provide the jump conditions across $r = a$. The application of a Kutta condition is still needed and we find that the velocity and total tangential stress are continuous across $r = a$. The matching must be done directly with the solutions of (7.6) and the similar equations, but this occasions no difficulty. It is now obvious that there was in fact no need to introduce the $\frac{1}{4}$ and $\frac{2}{7}$ layers of the previous analysis, except as a matter of analytical convenience, for these layers are contained in (7.6) and the similar equations. However, from a practical standpoint the ordinary differential equation (7.6) is hard to solve in closed form, and except for the simplest geometries one must either satisfy (7.1) and (7.2), or use numerical work.

As an example of the effect of large plate gap, we shall consider the two-dimensional problem of a strip rising with velocity U between rigid horizontal walls $z = h_T, -h_B$. With Cartesian coordinates (x, y, z) the body is $-a < x < a$, $z = 0$, and all velocities are independent of y. The velocities parallel to the Cartesian axes are (u, v, w). Then outside thin shear layers on $x = \pm a$,

$$v = V(x), \quad u = \frac{\nu}{2\Omega}\frac{d^2V}{dx^2}, \quad w = -\frac{\nu z}{2\Omega}\frac{d^3V}{dx^3}+W(x). \tag{7.7}$$

Application of the Ekman suction relations gives

$$V_T(x) = -\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}} U\left[x - A_T \frac{\sinh(p_T x/\nu^{\frac{1}{4}})}{\sinh(p_T a/\nu^{\frac{1}{4}})}\right] \quad (z>0, \quad |x|<a), \tag{7.8}$$

$$V_B(x) = \left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}} U\left[x - A_B \frac{\sinh(p_B x/\nu^{\frac{1}{4}})}{\sinh(p_B a/\nu^{\frac{1}{4}})}\right] \quad (z<0, \quad |x|<a), \tag{7.9}$$

$$V_E(x) = \left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}} UB\, \mathrm{e}^{-p(x-a)/\nu^{\frac{1}{4}}} \quad (-h_B < z < h_T, \quad x>a), \tag{7.10}$$

where
$$p_T^2 = 2\Omega^{\frac{1}{2}}/h_T, \quad p_B^2 = 2\Omega^{\frac{1}{2}}/h_B, \quad p^2 = 2\Omega^{\frac{1}{2}}/h. \tag{7.11}$$

We have assumed that the 'swirl' velocity is odd in x. (Actually, this would follow from the jump conditions with a little labour.)

From the continuity of V and total tangential stress across $x = a$, we find

$$B = \frac{\nu^{\frac{1}{4}}(h_T - h_B) + ap_B h_B \coth(p_B a/\nu^{\frac{1}{4}}) - ap_T h_T \coth(p_T a/\nu^{\frac{1}{4}})}{ph + p_T h_T \coth(p_T a/\nu^{\frac{1}{4}}) + p_B h_B \coth(p_B a/\nu^{\frac{1}{4}})} = a - A_B = A_T - a. \tag{7.12}$$

Expanding these solutions about $x = a$ gives, as before, matching conditions for the $\frac{1}{3}$ layer solution and it is clear that the $\frac{1}{3}$ layer structure is identical with that obtained earlier for the rising disk.

It is of interest to calculate the drag. For small Ekman number, the dominant contribution is from the pressure†, which is given (everywhere) by

$$-2\Omega\rho v = -\partial p/\partial x. \tag{7.13}$$

An elementary calculation gives

$$D = 4\rho\Omega U\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}}\left[\frac{2a^3}{3} - \frac{A_B a\nu^{\frac{1}{4}}}{p_B}\coth(p_B a/\nu^{\frac{1}{4}}) - \frac{A_T a\nu^{\frac{1}{4}}}{p_T}\coth(p_T a/\nu^{\frac{1}{4}}) + \frac{A_B \nu^{\frac{1}{2}}}{p_B^2} + \frac{A_T \nu^{\frac{1}{2}}}{p_T^{21}}\right]. \tag{7.14}$$

In the special case when $h_T = h_B$ (7.14) becomes

$$D = 4\rho U\Omega\left(\frac{\Omega}{\nu}\right)^{\frac{1}{2}} a^3\left[\frac{2}{3} - \left(\frac{Eh^2}{a^2}\right)^{\frac{1}{4}}\coth 2\left(\frac{Eh^2}{a^2}\right)^{-\frac{1}{4}} + \frac{1}{2}\left(\frac{Eh^2}{a^2}\right)^{\frac{1}{2}}\right]. \tag{7.15}$$

The drag decreases as the relative thickness $(Eh^2/a^2)^{\frac{1}{4}}$ of the $\frac{1}{4}$ layer increases. This calculation is valid provided the distance between the end walls is not so large that the $\frac{1}{3}$ layer becomes thick, i.e. provided (7.4) is satisfied.

When h is so large that

$$E^{-1} \gg \frac{h_T}{a} \gg E^{-\frac{1}{2}}, \quad E^{-1} \gg \frac{h_B}{a} \gg E^{-\frac{1}{2}}, \tag{7.16}$$

the swirl velocities become $o(\nu^{-\frac{1}{2}})$ and the drag given by (7.14) is

$$D = \frac{8}{45}\frac{\rho\Omega^2 U a^5}{\nu}\left(\frac{1}{h_B} + \frac{1}{h_T}\right). \tag{7.17}$$

† The contribution from the normal viscous stresses is $O(\nu^{\frac{1}{2}})$.

8. Unbounded flow

We find that we cannot determine the flow when h/a is comparable with E^{-1}. However, when h/a becomes large compared with E^{-1}, we find below that the end surfaces can be neglected and a Taylor column of finite length appears. The Taylor column produced by a disk moving steadily in an unbounded slightly viscous fluid was found by Morrison & Morgan (1956). For the purpose of comparison with previous work, we shall now rederive their solution in a more direct way.†

Outside the Ekman layer on the body, the z-derivatives in the viscous stresses are negligible and the (linearized) equations of motion are

$$-2\Omega v = -\frac{1}{\rho}\frac{\partial p}{\partial r}, \tag{8.1}$$

$$2\Omega u = \nu\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}\right), \tag{8.2}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right), \tag{8.3}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ur)+\frac{\partial w}{\partial z} = 0. \tag{8.4}$$

We have dropped the viscous term in the radial momentum equation (8.1), because an argument similar to that in §2 shows that this term is negligible compared with the Coriolis term if the thickness of the shear layer is large compared with the width of the Ekman layer. We cannot, however, make the boundary-layer approximation in the other viscous terms, replacing $\partial^2 v/\partial r^2+(1/r)\,\partial v/\partial r-v/r^2$ by just $\partial^2 v/\partial r^2$, because the shear layers in which viscosity is important will fatten away from the body where viscosity damps out the disturbance and their horizontal scale will become comparable with the body size. In other words, treating the shear layers as thin surfaces will not give a uniformly valid solution. The approximations in (8.1) to (8.4) can be checked *a posteriori*.

The boundary conditions are

$$u\to 0,\quad v\to 0,\quad w\to 0 \quad \text{as} \quad r^2+z^2\to\infty, \tag{8.5}$$

together with the Ekman compatibility relation

$$w-U = \pm\frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}\frac{1}{r}\frac{\partial}{\partial r}(rv) \quad \text{on} \quad r<a \quad (z=\pm 0), \tag{8.6}$$

where U is the vertical velocity of the disk. By inspection, it is clear that u and v are odd functions of z, and w is even. We consider the solution for $z>0$. In particular

$$v = 0 \quad (z=0,\quad r>a). \tag{8.7}$$

It is immediate that

$$v = \int_0^\infty A(k)\,J_1(kr)\,e^{-\nu k^3 z/2\Omega}\,dk, \tag{8.8}$$

$$w = -\int_0^\infty A(k)\,J_0(kr)\,e^{-\nu k^3 z/2\Omega}\,dk, \tag{8.9}$$

$$u = -\frac{\nu}{2\Omega}\int_0^\infty k^2 A(k)\,J_1(kr)\,e^{-\nu k^3 z/2\Omega}\,dk, \tag{8.10}$$

† A similar analysis has been given by W. S. Childress (unpublished).

give a solution of the equations. From the boundary conditions (8.6) and (8.7), we obtain the dual integral equations

$$\int_0^\infty A(k)\left[1-\frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{\frac{1}{2}}k\right]J_0(kr)\,\mathrm{d}k = -U \quad (r<a), \tag{8.11}$$

$$\int_0^\infty A(k)\,J_1(kr)\,\mathrm{d}k = 0 \quad (r>a). \tag{8.12}$$

For $E^{\frac{1}{2}} \ll 1$, the square bracket in (8.11) can be replaced by unity† and we have the well known solution

$$A(k) = \frac{2Ua}{\pi}\left(\cos ka - \frac{\sin ka}{ka}\right). \tag{8.13}$$

Substitution into (8.8), (8.9) and (8.10) gives the solution of Morrison & Morgan.

For $|r-a| \gg (\nu z/2\Omega)^{\frac{1}{3}}$, we have a Taylor column with velocities

$$w = U, \quad v = -\frac{2U}{\pi}\frac{r}{(a^2-r^2)^{\frac{1}{2}}} \quad (r<a), \tag{8.14}$$

$$w = \frac{2U}{\pi}\left\{\sin^{-1}\frac{a}{r} - \frac{a}{(r^2-a^2)^{\frac{1}{2}}}\right\} \quad (v=0 \quad r>a). \tag{8.15}$$

Note that

$$\int_0^\infty rw\,\mathrm{d}r = 0,$$

so that there is no net mass flow across any horizontal plane. The jumps in velocity across $r=a$ are smoothed out in the shear layer of thickness

$$\delta \sim (\nu z/2\Omega)^{\frac{1}{3}}. \tag{8.16}$$

When $z \sim a/E$, the shear layer is comparable to the body in horizontal dimension and the velocities start decaying to zero.

To determine the structure of the shear layers for $z \ll a/E$, we replace the integrand by its asymptotic value for large argument. The justification is that as $r \to a$ the integrand loses its oscillatory character and the integral is dominated by contributions from large k. Then

$$\begin{aligned} w &\sim -\frac{2Ua}{\pi}\int_0^\infty \cos ka\left(\frac{2}{\pi kr}\right)^{\frac{1}{2}}\cos\left(kr-\frac{\pi}{4}\right)\mathrm{e}^{-\nu k^3 z/2\Omega}\,\mathrm{d}k \\ &\sim -\left(\frac{a}{\pi^3}\right)^{\frac{1}{2}}\left(\frac{2\Omega}{\nu z}\right)^{\frac{1}{6}}U\int_0^\infty \mathrm{e}^{-p^3}(\cos p\zeta+\sin p\zeta)\frac{\mathrm{d}p}{p^{\frac{1}{2}}}, \end{aligned} \tag{8.17}$$

where

$$\zeta = \frac{r-a}{z^{\frac{1}{3}}}\left(\frac{2\Omega}{\nu}\right)^{\frac{1}{3}}. \tag{8.18}$$

Thus the shear layer contains velocities $O(\nu^{-\frac{1}{6}})$. For large positive ζ, (8.17) becomes

$$w \sim -\left(\frac{a}{\pi^3}\right)^{\frac{1}{2}}\left(\frac{2\Omega}{\nu z}\right)^{\frac{1}{6}}\frac{U}{\zeta^{\frac{1}{2}}}\int_0^\infty(\cos p+\sin p)\frac{\mathrm{d}p}{p^{\frac{1}{2}}} = -\frac{2^{\frac{1}{2}}U}{\pi}\frac{a^{\frac{1}{2}}}{(r-a)^{\frac{1}{2}}}, \tag{8.19}$$

which matches the asymptotic form of the velocity outside the Taylor column. For large negative ζ, the expression (8.17) tends to zero to this order, as it must since the velocity

† The effect of the neglected term can be found by placing it on the right-hand side of (8.11) and inserting (8.13) for $A(k)$. This is equivalent to calculating the higher order flow induced by the weak Ekman suction as done by W. S. Childress (unpublished).

inside the column is $o(\nu^{-\frac{1}{6}})$. Similar results hold for v. The shear-layer structure corresponds to the similarity solution of § 3 with $m = -\frac{1}{6}$.

The drag on the body is readily found to have the value

$$D = \tfrac{16}{3}\rho U\Omega a^3 \tag{8.20}$$

when equation (8.1) is integrated to give the pressure using the velocity field given by (8.14). Note that in contrast with the bounded case, the drag is independent of the viscosity. It agrees with value found by Grace (1926) and Stewartson (1952) by an unsteady inviscid analysis. The swirl velocity is $O(1)$ inside the column, as compared with $O(\nu^{-\frac{1}{2}})$ for the bounded case, and the Ekman layer is therefore weak. Thus to leading order, the boundary condition (8.6) could have been replaced by the simpler condition $w = U$. It is clear now that the solution obtained for the disk holds for an arbitrary body of revolution (provided its length is not comparable with a/E), and that the drag is likewise independent of the shape.

It is noteworthy that the flow field near the body is independent of viscosity, outside the shear layers, and yet depends on the correct form of the viscous stresses being employed. Thus if we had made the boundary-layer approximation to the viscous stresses and the equation of continuity, we would have found a Taylor column with velocities

$$\left.\begin{aligned} v &= -\frac{Ur}{(a^2-r^2)^{\frac{1}{2}}}, \quad w = U \quad (r < a), \\ v &= 0, \quad w = -\frac{Ua^2}{(r^2-a^2)^{\frac{1}{2}}[r+(r^2-a^2)^{\frac{1}{2}}]} \quad (r > a). \end{aligned}\right\} \tag{8.21}$$

The drag would then be greater by a factor $\frac{1}{2}\pi$ than the true value. The reason for the qualitative difference is as follows. The Taylor–Proudman theorem still holds as an approximation, but the geostrophic flow is not constrained by Ekman layers on end walls as in the bounded case but by the region where the shear layer fattens and disappears. This is where the fluid picks up its 'swirl' and this region must be described properly and not by the boundary-layer approximation. It is indeed strange that the structure of the shear layer near the disk cannot be found by a local analysis, but the strong vertical constraint imposed by the rotation is responsible.

Appendix 1. The Wiener–Hopf factorization

The analysis leading to equation (5.51) is now outlined. Consider the infinite product

$$G(z) = \prod_{n=1}^{\infty}\left(1+\frac{z}{n^{\frac{1}{3}}}\right)\exp\left(-\frac{z}{n^{\frac{1}{3}}}+\frac{z^2}{2n^{\frac{2}{3}}}-\frac{z^3}{3n}\right). \tag{A 1}$$

It is easily shown that the product converges. The function $G(z)$ is entire with zeros on the negative real axis and $G(0) = 1$. By virtue of the formula

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right), \tag{A 2}$$

it follows that

$$\begin{aligned} \frac{\sin \pi z^3}{\pi z^3} &= [G(z)\,G(\omega z)\,G(-\omega^2 z)]\,[G(-z)\,G(-\omega z)\,G(\omega^2 z)] \\ &= H(z)\,H(-z), \quad \text{say} \end{aligned} \tag{A 3}$$

where $\omega = e^{2\pi i/3}$ and $\omega^2 = e^{-2\pi i/3}$ are cube roots of unity. The function $H(z)$ is entire and has its zeros in $\frac{7}{6}\pi > \arg z > \frac{1}{6}\pi$; this region corresponds to the upper half of the α-plane.

The behaviour of $G(z)$ as $z \to \infty$ is required. We start with the result that

$$-\frac{1}{z^2}\frac{d}{dz}\ln G(z) = \sum_1^\infty \frac{z}{n(z+n^{\frac{1}{3}})} \sim 3\ln z + \sum_0^\infty \frac{b_n}{z^n} \quad (-\pi < \arg z < \pi), \tag{A 4}$$

where the b_n are numbers that can be expressed as values of the Riemann Zeta-function. This result can be proved by expressing the series, with the aid of the Euler–Plana formula (Whittaker & Watson 1950, p. 145), in terms of integrals and then finding asymptotic forms of the integrals. The argument is long and will be omitted. For reference, we note that

$$b_0 = \gamma \quad b_n = (-1)^n \zeta(1-\tfrac{1}{3}n), \tag{A 5}$$

and in particular $b_3 = \frac{1}{2}$. [These values can be obtained formally by the use of divergent series.]

Integrating (A 4), we find for $-\pi < \arg z < \pi$,

$$\ln G(z) \sim -z^3 \ln z + \tfrac{1}{3}z^3(1-\gamma) - \tfrac{1}{2}b_1 z^2 - b_2 z - b_3 \ln z + A + \sum_4^\infty \frac{b_n}{(n-3)\,z^{n-3}}. \tag{A 6}$$

To determine the constant A, we note that, from (A 1) and the infinite product expression for $\Gamma(z)$,

$$\ln G(z) + \ln G(\omega z) + \ln G(\omega^2 z) = -3\ln z - \gamma z^3 - \ln \Gamma(z^3)$$
$$\sim -3z^3\ln z + (1-\gamma)\,z^3 - \tfrac{3}{2}\ln z - \tfrac{1}{2}\ln(2\pi) - \sum_{n=1}^\infty \frac{(-1)^{n-1}B_n}{2n(2n-1)\,z^{6n-3}}, \tag{A 7}$$

for $|\arg z| < \frac{1}{3}\pi$. Comparing (A 6) and (A 7), we find all terms agree and that

$$A = -\tfrac{1}{6}\ln(2\pi). \tag{A 8}$$

To remove the non-algebraic behaviour of $G(z)$ at ∞, we define

$$\hat{G}(z) = G(z)/\exp\,[-z^3\ln z + \tfrac{1}{3}z^3(1-\gamma) - \tfrac{1}{2}b_1 z^3 - b_2 z], \tag{A 9}$$

with the plane cut along $\arg z = \pi$. Then

$$\hat{G}(z) \sim \frac{1}{(2\pi)^{\frac{1}{6}} z^{\frac{1}{2}}} \exp\left[\sum_4^\infty \frac{b_n}{(n-3)\,z^{n-3}}\right] \quad (|\arg z| < \pi), \tag{A 10}$$

and $\hat{G}(0) = 1$.

We define the function

$$\hat{H}(z) = \hat{G}(z)\,\hat{G}(\omega z)\,\hat{G}(-\omega^2 z). \tag{A 11}$$

Then this function is analytic and without zeros in $-\pi < \arg z < \frac{1}{3}\pi$, and to leading order

$$\hat{H}(z) \sim \frac{-i}{(2\pi)^{\frac{1}{2}}\,z^{\frac{3}{2}}} \tag{A 12}$$

as $z \to \infty$ in this sector. Similarly, $\hat{H}(-z)$ is analytic and non-zero in $0 < \arg z < \frac{4}{3}\pi$, and

$$\hat{H}(-z) \sim \frac{-i}{(2\pi)^{\frac{1}{2}}\,(-z)^{\frac{3}{2}}} \tag{A 13}$$

in this sector. Paying careful attention to the argument of $\ln z$, we find that

$$\hat{H}(z)\,\hat{H}(-z) = e^{\pi i z^3}\,(\sin \pi z^3)/\pi z^3. \tag{A 14}$$

The factorization of (5.49) into 'plus' and 'minus' functions is now straightforward on putting $z \propto \alpha\, e^{i\pi/6}$. Since $\hat{H}(0) = 1$, we can ensure, by choice of the constant factor, that $S_-(0) = 1$ and $S_+(0) = 1$.

References

Baker, D. J. 1967 *J. Fluid Mech.* **29**, 165.
Carrier, G. F. 1964 *Proc. 11th Congr. Appl. Mech.* (*Munich*), p. 69.
Grace, S. F. 1926 *Proc. Roy. Soc.* A **113**, 46.
Greenspan, H. P. & Howard, L. N. 1963 *J. Fluid Mech.* **17**, 385.
Hide, R. & Titman, C. W. 1967 *J. Fluid Mech.* **29**, 39.
Jacobs, S. J. 1964 *J. Fluid Mech.* **20**, 581.
Maxworthy, A. 1968 *J. Fluid Mech.* **31**, 643.
Moore, D. W. & Saffman, P. G. 1968 *J. Fluid Mech.* **31**, 635.
Morrison, J. A. & Morgan, G. W. 1956 *Tech. Rep. Div. Appl. Math., Brown University*, no. 8.
Proudman, I. 1956 *J. Fluid Mech.* **1**, 505.
Stewartson, K. 1952 *Proc. Camb. Phil. Soc.* **48**, 168.
Stewartson, K. 1957 *J. Fluid Mech.* **3**, 17.
Stewartson, K. 1966 *J. Fluid Mech.* **26**, 131.
Whittaker, E. T. & Watson, G. N. 1950 *Modern analysis*, Cambridge University Press.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES